ON SOME RESULTS INVOLVING MANY VARIABLE FUNCTIONS OF HARDY INTEGRAL INEQUALITIES

ABSTRACT. Inequalities involving superquadratic and subquadratic functions of independent interest are established for refined Hardy-type inequalities using fairly elementary analytical ideas. The consequences of our main results are derived and the corresponding proofs are presented.

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1. INTRODUCTION

In 1920, Hardy announced his well celebrated and famous inequality [6]. If $p > 1, f \geq 0$, $p-$integrable on $(0, \infty)$ and

$$F(x) = \int_0^x f(t)dt, \text{then,}$$

$$\int_0^\infty \left(\frac{F(t)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \; p > 1 \tag{1.1}$$

where f is a non-negative measurable function and the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. The inequality was proved in [8] and also holds for:

$$\int_a^b \left(\frac{F(t)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f^p(x)dx$$

where $0 < a < b < \infty$, see [7].
An improved form of the inequality was observed in [2], where f is non-decreasing. If $f \geq 0$, and non-decreasing, F is as defined by (1.1). $f \geq 0, \; g > 0, \; \frac{x}{g(x)}$ is non-increasing, $p > 1$ and $0 < a < 1$ then

$$\int_0^\infty \left(\frac{F(t)}{g(x)}\right)^p dx \leq \frac{p}{a(1-p)(1-a)^{p-1}} \int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^p dx \tag{1.2}$$

The inequality has been developed and applied in almost unbelievable ways, see [10 - 14] and the references therein. The prehistory of the inequality could be sourced in [9]. This work is motivated by [15] which used adaptation of convexity of a function to obtain the following: For $p \geq 1$ the refined inequality (1.4).

$$\int_0^\infty g(x)^{-k} F^p(x)dg(x) + \left(\frac{p}{k-1}\right)^p g(b)^{1-k} F^p(b) \leq \left(\frac{p}{k-1}\right)^p \int_0^\infty g(x)^{p-k} f^p(x)dg(x) \tag{1.3}$$

and

$$\int_b^\infty g(x)^{-k} F^p(x)dg(x) + \left(\frac{p}{1-k}\right)^p g(b)^{1-k} F^p(b) \leq \left(\frac{p}{1-k}\right)^p \int_b^\infty g(x)^{p-k} f^p(x)dg(x) \tag{1.4}$$

hold with both inequalities reversed in $0 < p \leq 1$. Equality holds in either inequality, when either $p = 1$ or $f = 0$. The constant $\left(\frac{p}{k-1}\right)^p$ or $\left(\frac{p}{1-k}\right)^p$ is the best possible when the left side of (1.5) or

(1.6) is unchanged, respectively. The result generalized [16].
The objective of this paper is to obtain a new integral inequality which is an extension of (1.1). Indeed, we shall show that (1.1) in its modified form leads us to some extensions, and a new generalization of a class of inequalities which are of Hardy-type integral inequalities.

2. Some Useful Definitions:

We need the following tools in the proofs of our main results.

Jensen inequality:
Let μ be a probability measure and let $\Phi \geq 0$ be a convex function. Then, for all $\zeta(x)$ be a integrable function we have

$$\int \Phi^{o}\zeta(\mu)d\mu \leq \Phi\left(\int \zeta(\mu)d\mu\right) \tag{2.1}$$

Chebyshev integral inequality:

If $\zeta, \omega : [\alpha, \beta] \longrightarrow \mathbb{R}$ are integrable functions, both increasing or both decreasing, and $\rho : [\alpha, \beta] \longrightarrow \mathbb{R}$ is a positive integrable function, then

$$\int_{\alpha}^{\beta} \rho(x)\zeta(x)dx \int_{\alpha}^{\beta} \rho(x)\omega(x)dx \leq \int_{\alpha}^{\beta} \rho(x)dx \int_{\alpha}^{\beta} \rho(x)\zeta(x)\omega(x)dx \tag{2.2}$$

We observe that if one of the functions ζ or ω is decreasing and the other is increasing, then (2.1) is reversed and where $\rho(x) = 1$. we have

$$\int_{\alpha}^{\beta} \rho(x)\zeta(x)dx \int_{\alpha}^{\beta} \rho(x)\omega(x)dx \leq \int_{\alpha}^{\beta} \rho(x)dx \int_{\alpha}^{\beta} \zeta(x)\omega(x)dx \tag{2.3}$$

Submultiplicative:

Let $\Phi \geq 0$ is submultiplicative, and $\Phi(0) = 0$. If $\Phi'(x)$ is non-decreasing (nonincreasing), then $\frac{\Phi(0)}{x}$ is non-decreasing (non-increasing).
A function Φ is called submultiplicative, if $\Phi(xy) \leq \Phi(x)\Phi(y)$, for all $x, y > 0$ In particular, for all $n \geq 1$, we have $\Phi(x^n) \leq \Phi^n(x)x > 0$.
A function $\varphi : [0, \infty) \longrightarrow \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$\Phi(y) - \Phi(x) - \Phi(|y - x|) \geq C_x(y - x), \ \forall\ y \geq 0 \tag{2.4}$$

We say that Φ is subquadratic if -Φ is superquadratic.

Corrolary 2.1: (See [17, Theorem 2.3].)
Let (Ω, μ) be a probability measure space. The inequality

$$\Phi\left(\int_{\Omega} f(s)d\mu(s)\right) \leq \int_{\Omega} \Phi(f(s))d\mu(s) - \int_{\Omega} \Phi\left(\left|f(s) - \int_{\Omega} f(s)d\mu(s)\right|\right) d\mu(s) \tag{2.5}$$

holds for all probability measures μ and all non negative μ-integrable functions f if and only if Φ is superquadratic. Moreover, (1.1) holds in the reversed direction if and only if Φ is subquadratic.

Corrolary 2.2: (See [17, Lemma 3.1].) $\Phi : [0, \infty) \longrightarrow \mathbb{R}$ is continuously differentiable and $\Phi(0) \leq 0$. If Φ' is superadditive or $\frac{\Phi'(x)}{x}$ is nondecreasing, then Φ is superquadratic.
Corollary 2.3

Let $a, b \in \mathbb{R}$. Suppose $\eta : \mathbb{R} \longrightarrow [0, \infty)$ is rd-continuous and $\phi : [0, \infty) \longrightarrow \mathbb{R}$ is a continuous, convex and superquadratic function. Then,

$$\left(\frac{1}{b-a}\int_a^b \eta(t)d\mu(t)\right)^r \leq \frac{1}{b-a}\int_a^b \left(\eta(x)^r - \left|\eta(x)^r - \frac{1}{b-a}\int_a^b \eta(t)d\mu(t)\right|^r\right) d\mu(x) \tag{2.6}$$

The proof of the next theorem is sufficient for Corollary 2.3.

Corollary 2.4

Let $u, \upsilon \in \mathbb{R}$ be non negative functions such as $\mu-$integral $\int_\tau^b \frac{u(x)\eta(x)}{(b-a)\sigma(x)-a} d\mu(x) < \infty$ and define the weight function $\upsilon(\tau)$ by

$$\upsilon(t) = (t-a)\int_t^b \frac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)} d\mu(x) \ t \in (a, b)$$

(i) If real valued function Φ is convex and superquadratic function such that on (a, b), $0 < a < b < \infty$ then,

$$\begin{aligned} &\int_a^b u(x)\eta(x)\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} \eta(t)d\mu(t)\right)^r \frac{\Delta x}{(x-a)} \\ &+ \int_a^b\int_t^b \left|(\eta(t))^r - \left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} \eta(t)d\mu(t)\right|^r\right) \frac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)} d\mu(x)d\mu(t). \\ &\leq \int_a^b u(x)\eta(x)(\eta(x))^q \frac{\Delta x}{(x-a)} \end{aligned} \tag{2.7}$$

holds for all $\mu-$integrable function $\eta \in \mathbb{R}$ such that $\eta(x) \in (a, c)$.

(ii) If the real valued function Φ is convex and superquadratic function such that on (a, b), $0 < a < c \leq \infty$ then, (2.7) holds in the reverse direction.

Proof:

(i) Applying Jensen′s inequality and Fubini theorem (See [17, Theorem 2.1].), we have

$$\begin{aligned} &\int_a^b u(x)\eta(x)\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} \eta(t)d\mu(t)\right)^r \frac{d\mu(x)}{(x-a)} \\ &\leq \int_a^b \frac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)}\int_a^{\sigma(x)} (\eta(t))^r d\mu(x)d\mu(t) \\ &- \int_a^b \frac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)}\int_t^b \left|(\eta(t))^r - \frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} \eta(t)d\mu(t)\right|^q d\mu(x) \\ &= \int_a^b \eta(t)^r \int_t^b \frac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)} d\mu(x)d\mu(t) \\ &- \int_a^b\int_t^b \left|(\eta(t))^r - \frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} \eta(t)d\mu(t)\right|^r \frac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)} d\mu(x)d\mu(t) \\ &= \int_a^b u(x)\eta(x)\eta(t)^q \frac{d\mu(t)}{(t-a)} \\ &- \int_a^b\int_t^b \left|(\eta(t))^r - \frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} \eta(t)d\mu(t)\right|^r \frac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)} d\mu(x)d\mu(t) \end{aligned} \tag{2.8}$$

(ii) This is similar to the proof of (i) above but the only difference is that in this case the inequality sign is reversed. The proof is complete.

Further simplification of (2.8), if $u(x) = (x-a), a < b < \infty$ and $u(x) = 1$ yields

$$\begin{aligned}
&\int_a^b \eta(x)\left(\frac{1}{(\sigma(x)-a)}\int_a^b \eta(t)\right) d\mu(x) - \int_a^b \eta(x)\eta(x)^r d\mu(x) \\
&\leq \int_a^b \eta(x)\eta(x)^r d\mu(x) \\
&- \int_a^b \int_t^b \left(\left|(\eta(t)) - \left(\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} \eta(t) d\mu(t)\right)\right|\right) \frac{\eta(x)}{(\sigma(x)-a)} d\mu(x) d\mu(t)
\end{aligned} \tag{2.9}$$

but if $b = \infty$ then, above inequality becomes

$$\begin{aligned}
&\int_a^\infty \eta(x)\left(\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} \eta(t)\right)^r \frac{d\mu(x)}{x-a} \\
&\leq \int_a^\infty (\eta(t)) \frac{\eta(x)^r}{(x-a)} d\mu(x) \\
&- \int_a^\infty \int_t^\infty \left|\eta(x) - \frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} \eta(t) d\mu(t)\right|^r \frac{\eta(x)}{(\sigma(x)-a)} d\mu(x) d\mu(t)
\end{aligned} \tag{2.10}$$

that is

$$\begin{aligned}
&\int_a^\infty \eta(x)\left(\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} \eta(t)\right)^r \frac{d\mu(x)}{x-a} - \int_a^\infty (\eta(t)) \frac{\eta(x)^r}{(x-a)} d\mu(x) \\
&\leq \int_a^\infty \int_t^\infty \left|\eta(x) - \frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} \eta(t) d\mu(t)\right|^r \frac{\eta(x)}{(\sigma(x)-a)} d\mu(x) d\mu(t)
\end{aligned} \tag{2.11}$$

(ii) The inequalities (2.11) hold in the reversed direction if Φ is subquadratic.

Corollary 2.5 :
Let $\omega_1(t),\ldots,\omega_i(t) > 0$, and $W(x) = \int_0^x \omega_1(t),\ldots,\omega_i(t)\ d_1t,\ldots,d_it$. If $\Phi'(x)$ is non-decreasing (non-increasing) then, function $\frac{\omega_1(x),\ldots,\omega_i(x)}{x^2}$ is also non-decreasing (non-increasing) on $(0,\infty)$. See [17] for the proof.

Corollary 2.6 :
Let $0 < b \le \infty, u : (0,\infty) \longrightarrow \mathbb{R}$ be a nonnegative weight function such that the function $x \longrightarrow \frac{u_1(x)\ldots u_i(x)}{x^2}$ is locally integrable on $(0,\infty)$, and define the weight function ν by

$$\nu_1(t)\ldots\nu_i(t) = t\int_t^b \frac{u_1(x)\ldots u_i(x)}{x^2}\ dx_1\ldots dx_i,\ \text{for all } t \in (0,b)$$

(1) If the real-valued function Φ is superquadratic on $(a,c), 0 \le a < c \le \infty$, then

(2.12)

$$\int_a^b u_1(x)\ldots u_i(x)\left(\frac{1}{x}\int_0^x \zeta_1(t)\ldots\zeta_i(t)d_1(t)\ldots d_i(t)\right)^r \frac{d_1(x)\ldots d_i(x)}{x} +$$
$$+\int_a^b\int_t^b\left(\zeta_1(t)\ldots\zeta_i(t) - \frac{1}{x}\int_0^x \zeta_1(t)\ldots\zeta_i(t)d_1(t)\ldots d_i(t)\right)^r \frac{u_1(x)\ldots u_i(x)}{x^2} d_1(x)\ldots d_i(x)\ d_1(t)\ldots d_i(t)$$
$$\le \int_a^b u_1(x)\ldots u_i(x)(\zeta_1(t)\ldots\zeta_i(t))^r\ \frac{d_1(x)\ldots d_i(x)}{x}\ \text{holds for all } \zeta \text{ with } a < \zeta(x) < c,\ 0 < x \le b$$

(2) If the real-valued function Φ is subquadratic on $(a,c), 0 \le a < c \le$, then (2.2) holds in the reversed direction.

Proof. We adopt the applications of refined Jensens inequality (2.1) and Fubinis theorem see [17]. we have that
(2.13)

$$\int_a^b u_1(x)\ldots u_i(x)\left(\frac{1}{x}\int_0^x \zeta_1(t)\ldots\zeta_i(t)d_1(t)\ldots d_i(t)\right)^r \frac{d_1(x)\ldots d_i(x)}{x}$$
$$\le \int_a^b \frac{u_1(x)\ldots u_i(x)}{x^2}\int_0^x (\zeta_1(t)\ldots\zeta_i(t))^r d_1(x)\ldots d_i(x)\ d_1(t)\ldots d_i(t) -$$
$$-\int_a^b \frac{u_1(x)\ldots u_i(x)}{x^2}\int_0^x\left(\zeta_1(t)\ldots\zeta_i(t) - \frac{1}{x}\int_0^x \zeta_1(t)\ldots\zeta_i(t)d_1(t)\ldots d_i(t)\right)^r \frac{u_1(x)\ldots u_i(x)}{x^2} d_1(x)\ldots d_i(x)\ d_1(t)\ldots d_i(t)$$
$$= \int_a^b (\zeta_1(t)\ldots\zeta_i(t))^r\int_t^b \frac{u_1(x)\ldots u_i(x)}{x^2} d_1(x)\ldots d_i(x) d_1(t)\ldots d_i(t) -$$
$$-\int_a^b\int_t^b\left(\zeta_1(t)\ldots\zeta_i(t) - \frac{1}{x}\int_0^x \zeta_1(t)\ldots\zeta_i(t)d_1(t)\ldots d_i(t)\right)^r \frac{u_1(x)\ldots u_i(x)}{x^2} d_1(x)\ldots d_i(x)\ d_1(t)\ldots d_i(t)$$
$$= \int_a^b \nu_1(t)\ldots\nu_i(t)(\zeta_1(t)\ldots\zeta_i(t))^r \frac{d_1(t)\ldots d_i(t)}{t} - \int_a^b\int_t^b\left(\zeta_1(t)\ldots\zeta_i(t)d_1(t)\ldots d_i(t) - \frac{1}{x}\int_0^x \zeta_1(t)\ldots\zeta_i(t)d_1(t)\ldots d_i(t)\right)^r$$
$$\times\ \frac{u_1(x)\ldots u_i(x)}{x^2} d_1(x)\ldots d_i(x)\ d_1(t)\ldots d_i(t)$$

which is the proof of the Corollary 2.6.
The proof of (2) is complete if the reserve sign of above inequality is case. However, is similar to the proof of (1).

Corollary 2.7 :
Let $0 < b \le \infty, u : (0,\infty) \longrightarrow \mathbb{R}$ be a non-negative locally integrable function $(0,\infty)$ and defined the function by

$$\xi_1(t)\ldots\xi_i(t) = \frac{1}{t}\int_b^t \frac{u_1(x)\ldots u_i(x)}{x^2}\ dx_1\ldots dx_i$$

(i) If the real-valued function Φ is superquadratic on $(a,c), 0 \leq a < c \leq \infty$, then

(2.14)

$$\int_b^\infty u_1(x)\dots u_i(x)\left(x\int_x^\infty \zeta_1(t)\dots\zeta_i(t)\frac{d_1(t)\dots d_i(t)}{t^2}\right)^r \frac{d_1(x)\dots d_i(x)}{x}+$$
$$+\int_b^\infty\int_b^t\left(\zeta_1(t)\dots\zeta_i - x\int_x^\infty \zeta_1(t)\dots\zeta_i(t)\frac{d_1(t)\dots d_i(t)}{t^2}\right)^r u_1(x)\dots u_i(x)d_1(x)\dots d_i(x)\ \frac{d_1(t)\dots d_i(t)}{t^2}$$
$$\leq \int_b^\infty \nu_1(x)\dots\nu_i(x)(\zeta_1(t)\dots\zeta_i(t))^r\ \frac{d_1(x)\dots d_i(x)}{x} \text{ holds for all } \zeta \text{ with } a<\zeta(x)<c, 0<x\leq b$$

Proof. With all assumption of the of the proof Corollary 2.6 we have that

(2.15)

$$\int_b^\infty u_1(x)\dots u_i(x)\left(x\int_x^0 \zeta_1(t)\dots\zeta_i(t)\frac{d_1(t)\dots d_i(t)}{t^2}\right)^r \frac{d_1(x)\dots d_i(x)}{x}$$
$$\leq \int_b^\infty u_1(x)\dots u_i(x)\int_x^0 (\zeta_1(t)\dots\zeta_i(t))^r\ \frac{d_1(t)\dots d_i(t)}{t^2}d_1(x)\dots d_i(x)-$$
$$-\int_b^\infty u_1(x)\dots u_i(x)\int_x^0\left(\zeta_1(t)\dots\zeta_i(t) - x\int_x^0 \zeta_1(t)\dots\zeta_i(t)\frac{d_1(t)\dots d_i(t)}{t^2}\right)^r\frac{d_1(t)\dots t_i(t)}{t^2}\ d_1(x)\dots d_i(x)$$
$$=\int_b^\infty (\zeta_1(t)\dots\zeta_i(t))^r\int_b^t u_1(x)\dots u_i(x)\ d_1(x)\dots d_i(x)\ d_1(t)\dots d_i(t)-$$
$$-\int_b^\infty\int_b^t\left(\zeta_1(t)\dots\zeta_i(t) - x\int_x^0 \zeta_1(t)\dots\zeta_i(t)d_1(t)\dots d_i(t)\right)^r u_1(x)\dots u_i(x)\ d_1(x)\dots d_i(x)\ d_1(t)\dots d_i(t).$$
$$\leq \int_b^\infty \xi_1(t)\dots\xi_i(t)(\zeta_1(t)\dots\zeta_i(t))^r\frac{d_1(t)\dots d_i(t)}{t} - \int_a^b\int_b^t\left(\zeta_1(t)\dots\zeta_i(t)d_1(t)\dots d_i(t) - x\int_x^0 \zeta_1(t)\dots\zeta_i(t)\frac{d_1(t)\dots d_i(t)}{t^2}\right)^r$$
$$\times\, u_1(x)\dots u_i(x)\ d_1(x)\dots d_i(x)\ \frac{d_1(t)\dots d_i(t)}{t^2}$$

which is the proof of the Corollary 2.7.

(ii) If the real-valued function Φ is subquadratic on $(a,c), 0 \leq a < c \leq$, then (2.2) holds in the reversed direction.

3. Some Extension of Hardy-type Inequalities

The following results are used as preliminary to our main results.

Theorem 3.1 :

Let $\zeta_1(x)\dots\zeta_i(x) \geq 0,\ \omega_1(x)\dots\omega_i(x) > 0,\ 0<a<1,\ r>1,\ q>\frac{r-a(r-1)}{2}$ and

$$W_1(x),\dots,W_i(x) = \int_0^x \omega_1(t),\dots,\omega_1(t)d_i(t),\dots,d_i(t) \text{ for all } i=1,2,3\dots\in\mathbb{N} \tag{3.1}$$

If the function $\frac{x}{\omega_1(x)\dots\omega_i(x)}$ is non-increasing function. Then the following inequality

(3.2)

$$\int_0^\infty \frac{F_1^r(x)\dots F_i^r(x)}{W_1^r(x)\dots W_i^r(x)}d_1(x),\dots,d_i(x) \leq \frac{1}{((b-1)(r-1)+2q-1)(1-a)^{r-1}}\int_0^\infty \frac{t^r\times(\zeta_1(t)\dots\zeta_i(t))^r}{W_1^r(t)\dots W_i^r(t)}d_1(t),\dots,d_i(t)$$

holds.

Proof.
(3.3)
$$\int_0^\infty \frac{F_1^r(x)\dots F_i^r(x)}{W_1^r(x)\dots W_i^r(x)} d_1(x),\dots,d_i(x) = \int_0^\infty W_1^{-r}(x)\dots W_i^{-r}(x)\left(t^{\frac{-a(r-1)}{r}} t^{\frac{a(r-1)}{r}} \zeta_1(t)\dots\zeta_i(t) d_i(t),\dots,d_i(t)\right)^r d_1(x),\dots,d_i(x)$$
$$\leq \int_0^\infty W_1^{-r}(x)\dots W_i^{-r}(x)\left(\left(\int_0^x t^a d_1(t),\dots,d_i(t)\right)^{\frac{p-1}{r}}\left(\int_0^x t^{a(r-1)}\times\zeta_1^r(t)\dots\zeta_i^{-r}(t) d_1(t),\dots,d_i(t)\right)^{\frac{1}{r}}\right)^r d_1(x),\dots,d_i(x)$$
$$= \int_0^\infty W_1^r(x)\dots W_i^r(x)\left(\int_0^x t^a d_1(t),\dots,d_i(t)\right)^{r-1}\int_0^x t^{a(r-1)}\times\zeta_1^r(t)\dots\zeta_i^r(t)\ d_1(t),\dots,d_i(t)\ d_1(x),\dots,d_i(x)$$
$$= \frac{1}{(1-a)^{r-1}}\int_0^x t^{a(r-1)}\times\zeta_1^r(t)\dots\zeta_i^r(t)\int_t^\infty x^{(1-a)(r-1)}\times W_1^{-r}(x)\dots W_i^{-r}(x) d_1(x),\dots,d_i(x)\ d_1(t),\dots,d_i(t)$$
$$\leq \frac{1}{(1-a)^{r-1}}\int_0^\infty t^{a(r-1)}\left(\frac{t^2}{W_1^{-r}(x)\dots W_i^{-r}(x)}\right)\zeta_1^r(t)\dots\zeta_i^r(t)\int_t^\infty x^{(1-a)(r-1)-2q} d_1(x),\dots,d_i(x)\ d_1(t),\dots,d_i(t)$$
$$\int_0^\infty \frac{F_1^r(x)\dots F_i^r(x)}{W_1^r(x)\dots W_i^r(x)} dx = \frac{1}{((b-1)(r-1)+2q-1)(1-a)^{r-1}}\int_0^\infty \frac{t^r\times(\zeta_1(t)\dots\zeta_i(t))^r}{W_1^p(t)\dots W_i^r(t)} d_1(x),\dots,d_i(x)\ d_1(t),\dots,d_i(t)$$
is valid. In particular, if we put $a=\frac{1}{r}$, $q=\frac{r}{2}$ and $W(x)=x$ we obtain (1.1).

We need the following inequality to proof the converse of the above Theorem 3.1.

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^r d_1(x),\dots,d_i(x) \geq \frac{1+r}{1-r}\left(\frac{r}{1+r}\right)^r\left(\frac{\zeta(x)}{x}\right)^r d_1(x),\dots,d_i(x) \tag{3.4}$$

Proof. Suppose $\frac{\omega_1(x),\dots,\omega_i(x)}{x^2}$ is non-increasing in Corollary 2.5 yields:
(3.5)
$$\int_0^\infty \frac{F_1^r(x)\dots F_i^r(x)}{W_1^r(x)\dots W_i^r(x)} d_1(x),\dots,d_i(x) = \int_0^\infty W_1^{-r}(x)\dots W_i^{-r}(x)\left(t^{\frac{-a(r-1)}{r}} t^{\frac{a(p-1)}{r}} \zeta_1(t)\dots\zeta_i(t) d_1(t),\dots,d_i(t)\right)^r d_1(x),\dots,d_i(x)$$
$$\leq \int_0^\infty W_1^{-r}(x)\dots W_i^{-r}(x)\left(\left(\int_0^x t^a\ d_1(t),\dots,d_i(t)\right)^{\frac{r-1}{r}}\left(\int_0^x t^{a(r-1)}\times\zeta_1^r(t)\dots\zeta_i^{-r}(t) d_1(t),\dots,d_i(t)\right)^{\frac{1}{r}}\right)^r d_1(x),\dots,d_i(x)$$
$$= \int_0^\infty W_1^r(x)\dots W_i^r(x)\left(\int_0^x t^a\ d_1(t),\dots,d_i(t)\right)^{r-1}\int_0^x t^{a(r-1)}\times\zeta_1^r(t)\dots\zeta_i^r(t)\ d_1(x),\dots,d_i(x)\ d_1(t),\dots,d_i(t)$$
$$= \frac{1}{(1+a)^{r-1}}\int_0^x t^{a(r-1)}\times\zeta_1^r(t)\dots\zeta_i^r(t)\int_t^\infty x^{(1-a)(r-1)}\times W_1^{-r}(x)\dots W_i^{-r}(x) d_1(x),\dots,d_i(x)\ d_1(t),\dots,d_i(t)$$
$$\leq \frac{1}{(1-a)^{r-1}}\int_0^\infty t^{a(r-1)}\left(\frac{t^2}{W_1^{-r}(x)\dots W_i^{-r}(x)}\right)\zeta_1^r(t)\dots\zeta_i^r(t)\int_t^\infty x^{(1-a)(r-1)-2r} d_1(x),\dots,d_i(x)\ d_1(t),\dots,d_i(t)$$
$$\int_0^\infty \frac{F_1^r(x)\dots F_i^r(x)}{W_1^r(x)\dots W_i^r(x)} d_1(x),\dots,d_i(x) =$$
$$= \frac{1}{((b+1)(r-1)+2q-1)(1-a)^{r-1}}\int_0^\infty \frac{t^r\times(\zeta_1(t)\dots\zeta_i(t))^r}{W_1^p(t)\dots W_i^r(t)} d_1(x),\dots,d_i(x)\ d_1(t),\dots,d_i(t)$$
If we put $a=\frac{1}{r}$, $q=\frac{r}{2}$ and $W_1(x),\dots,W_i(x)=x$ we obtain (1.2).

Theorem 3.2 :
Let $\zeta_1(x)\dots\zeta_i(x)\geq 0,\ \omega_1(x)\dots\omega_i(x)>0,\ 0<a<1,\ r>1,\ q>\frac{r-a(r-1)}{2}$,
$$W(x)=\int_0^x \omega(t)d_1(t),\dots,d_i(t) \text{ and } F(x)=\int_0^x \zeta(t)d_1(t),\dots,d_i(t)$$
Let $\Phi(x)$ and $\phi(x)$ be increasing, submultiplicative and convex functions such that $\Phi(\zeta(x))$ and $\phi(\omega(x))$ are integrable function, then the following holds:

$$\int_0^\infty \Phi(F_1(x)\dots F_i(x))\phi(W_1(x)\dots W_i(x))x^{1-r}d_1(x),\dots,d_i(x) \le \le \frac{1}{1-r}\int_0^\infty \Phi(\zeta_1(x)\dots\zeta_i(x))\phi(\omega_1(x)\dots\omega_i(x))x^{1-r}d_1(x),\dots,d_i(x) \tag{3.6}$$

for all $r < 1$.

Proof. If the functions Φ and ϕ are convex and submultiplicative, we obtain

(3.7)

$$\int_0^\infty \frac{\Phi(F_1(x)\dots F_i(x))\phi(W_1(x)\dots W_i(x))x^{1-r}}{\Phi(x)\phi(x)}d_1(x),\dots,d_i(x) =$$
$$= \int_0^\infty \frac{x^{1-r}}{\Phi(x)\phi(x)}\Phi\left(\int_0^x \zeta_1(t)\dots\zeta_i(t)d_1(t),\dots,d_i(t)\right)\phi\left(\int_0^x \omega_1(t)\dots\omega_i(t)d_1(t),\dots,d_i(t)\right)d_1(x),\dots,d_i(x)$$
$$\le \int_0^\infty x^{1-r}\Phi\left(\frac{1}{x}\int_0^x \zeta_1(t)\dots\zeta_i(t)d_1(t),\dots,d_i(t)\right)\phi\left(\frac{1}{x}\int_0^x \omega_1(t)\dots\omega_i(t)d_1(t),\dots,d_i(t)\right)d_1(x),\dots,d_i(x)$$
$$= \int_0^\infty x^{-r}\left(\int_0^x \Phi(\zeta_1(t)\dots\zeta_i(t))\phi(\zeta_1(t)\dots\zeta_i(t))d_1(t),\dots,d_i(t)\right)\phi\left(\int_0^x \omega_1(t)\dots\omega_i(t)d_1(t),\dots,d_i(t)\right)d_1(x),\dots,d_i(x)$$

In view (2.1), we have the following

$$\int_0^\infty \frac{\Phi(F_1(x)\dots F_i(x))\phi(W_1(x)\dots W_i(x))x^{1-r}}{\Phi(x)\phi(x)}d_1(x),\dots,d_i(x) \le \le \int_0^\infty x^{-r}\left(\int_0^x \Phi(\zeta_1(t)\dots\zeta_i(t))\phi(\omega_1(t)\dots\omega_i(t))dt\right)d_1(x),\dots,d_i(x) \tag{3.8}$$

Since functions Φ, ϕ, $\omega_1(x)\dots\omega_i(x)$ and $\zeta_1(x)\dots\zeta_i(x)$ are non-decreasing then the function $\Phi^o(\zeta_1(x)\dots\zeta_i(x))$ and $\varphi^o(\omega_1(x)\dots\omega_i(x))$ are also non-decreasing and consequently we can applied (2.3) where $\rho(x) = 1$, and by the inequality (3.8), we have

$$\begin{aligned} &\int_0^\infty \frac{\Phi(F_1(x)\dots F_i(x))\phi(W_1(x)\dots W_i(x))x^{1-r}}{\Phi(x)\phi(x)}d_1(x),\dots,d_i(x) \le \\ &\le \int_0^\infty \Phi(\zeta_1(t)\dots\zeta_i(t))\phi(\omega_1(t)\dots\omega_i(t))\left(\int_t^\infty x^{-r}\Phi(x)\phi(x)dx\right)d_1(t),\dots,d_i(t) \\ &= \int_0^\infty \Phi(\zeta_1(t)\dots\zeta_i(t))\phi(\omega_1(t)\dots\omega_i(t))\left(\int_t^\infty x^{-r}dx\right)d_1(t),\dots,d_i(t) \\ &\le \frac{1}{1-r}\int_0^\infty x^{1-r}\Phi(\zeta_1(t)\dots\zeta_i(t))\phi(\omega_1(t)\dots\omega_i(t))d_1(t),\dots,d_i(t) = \\ &= \frac{1}{1-r}\int_0^\infty x^{1-r}\prod_{n=1}^{i}\Phi(\zeta_i(t))\phi(\omega_i(t))d_1(t),\dots,d_i(t) \quad \square \end{aligned} \tag{3.9}$$

Theorem 3.2 :
Let $\zeta_1\dots\zeta_i \ge 0$, $F_1\dots F_i$ and $W_i\dots w_i$ are non-decreasing. Let $\omega_1\dots\omega_i > 0$, be continuous on $(0,\infty)$. Let $\Phi \ge 0$ and non-decreasing with $a < b \le \infty$. If $\omega_1\dots\omega_i$ is non-increasing and $\Phi(\frac{\zeta(x)}{\omega(x)})$ is integrable on $a < b \le \infty$, the following inequality is valid.

$$\int_a^b \Phi\left(\frac{F_1(x)\dots F_i(x)}{W_1(x)\dots W_i(x)}\right)d_1(x),\dots,d_i(x) \le \int_a^b \phi\left(\frac{\zeta_1(x)\dots\zeta_i(x)}{\omega_1(x)\dots\omega_i(x)}\right)d_1(x),\dots,d_i(x) \tag{3.10}$$

by convexity $\Phi(x) = x^r, r \ge 1$ and $\Phi = \phi$, the above inequality can be written in the form of

$$\int_a^b \left(\frac{F_1(x)\dots F_i(x)}{W_1(x)\dots W_i(x)}\right)^r d_1(x),\dots,d_i(x) \le \int_a^b \left(\frac{\zeta_1(x)\dots\zeta_i(x)}{\omega_1(x)\dots\omega_i(x)}\right)^r d_1(x),\dots,d_i(x) \tag{3.11}$$

Proof.

$$\int_a^b \Phi\left(\frac{F_1(x)\dots F_i(x)}{W_1(x)\dots W_i(x)}\right) d_1(x),\dots,d_i(x) \le \int_a^b \phi\left(\frac{\zeta_1(x)\dots\zeta_i(x)}{\omega_1(x)\dots\omega_i(x)}\right) d_1(x),\dots,d_i(x) \tag{3.12}$$

Since ω is non-increasing, Φ and ϕ are non-decreasing, we have
(3.13)

$$\int_a^b \Phi\left(\frac{F_1(x)\dots F_i(x)}{W_1(x)\dots W_i(x)}\right) d_1(x),\dots,d_i(x) = \int_a^b \Phi\left(\frac{1}{W_1(x)\dots W_i(x)}\int_0^x \zeta(t)d_1(t),\dots,d_i(t)\right)$$
$$\le \int_a^b \Phi\left(\frac{x\times(\zeta_1(x)\dots\zeta_i(x)}{\omega_1(x)\dots\omega_i(x)}\right) d_1(x),\dots,d_i(x) = \int_a^b \Phi\left(\frac{x\times(\zeta_1(x)\dots\zeta_i(x)}{\int_0^x(\omega_1(t)\dots\omega_i(t)})d_1(t),\dots,d_i(t)\right)$$
$$= \int_a^b \Phi\left(\frac{x\times(\zeta_1(x)\dots\zeta_i(x)}{\int_0^x(\omega_1(t)\dots\omega_i(t)d_1(t),\dots,d_i(t)})\right) d_1(x),\dots,d_i(x) \le \int_a^b \Phi\left(\frac{x\times(\zeta_1(x)\dots\zeta_i(x)}{(\omega_1(x)\dots\omega_i(x))}\right) d_1(x),\dots,d_i(x)$$

The next theorem is a generalization of Theorem 3.3 in [3].
Theorem 3.3:
Let $\Phi(x)\ge 0$ be a twice differentiable function on $(0,\infty)$, convex, submultiplicative and $\omega(0)=0$. Suppose $q\in\mathbb{N}$ and $r>1$. If $x^{2r}\frac{\Phi(\zeta_1(x)\dots\zeta_i(x))}{\Phi(x)}$ is integrable , then the following inequality:
(3.14)

$$\int_0^\infty x^{2-r}\frac{\Phi(x^q(F_1(x)\dots F_i(x)))}{\Phi^{r+2}(x)}d_1(x),\dots,d_i(x) \le \frac{1}{p-1}\int_0^\infty x^{2-r}\frac{\Phi(\zeta_1(x)\dots\zeta_i(x))}{\Phi(x)}d_1(x),\dots,d_i(x)$$

holds.

Proof.
(3.15)

$$\int_0^\infty x^{2-r}\frac{\Phi(x^q\times(F_1(x)\dots F_i(x)))}{\Phi^{r+2}(x)} = \int_0^\infty \frac{x^{2-r}}{\Phi^{r+2}(x)}\Phi\left(x^{q+1}\frac{(F_1(x)\dots F_i(x))}{x}\right) d_1(x),\dots,d_i(x)$$
$$\le \int_0^\infty \frac{x^{2-r}}{\Phi^{r+2}(x)}\Phi(x^{q+1})(x)\Phi\left(\frac{(F_1(x)\dots F_i(x))}{x}\right) d_1(x),\dots,d_i(x) \le \int_0^\infty \frac{x^{2-r}}{\Phi(x)}\Phi\left(\frac{(F_1(x)\dots F_i(x))}{x}dx\right) d_1(x),\dots,d_i(x)$$
$$= \int_0^\infty \frac{x^{2-pr}}{\Phi(x)}\Phi\left(\frac{1}{x}\int_0^x (F_1(x)\dots F_i(x))\right) d_1(x),\dots,d_i(x) =$$
$$= \int_0^\infty \Phi(\zeta_1(x)\dots\zeta_i(x))\left(\int_t^\infty x^{-r}\frac{x}{\Phi(x)}d_1(x),\dots,d_i(x)\right) d_1(t),\dots,d_i(t)$$
$$= \int_0^\infty \Phi(\zeta_1(x)\dots\zeta_i(x))\frac{t}{\Phi(x)}\left(\int_t^\infty x^{-p}d_1(x),\dots,d_i(x)\right) d_1(t),\dots,d_i(t) \le$$
$$\le \frac{1}{r-1}\int_0^\infty x^{2-r}\Phi(\zeta_1(x)\dots\zeta_i(x))d_1(x),\dots,d_i(x) \quad \square$$

Theorem 3.4
Let $0\le\zeta$, non-decreasing on $(0,\infty)$ and define F and W as above. If $0\le\omega$ is a non-decreasing on $(0,\infty)$.$\Phi\ge 0$ and non-decreasing with $0<a<b<\infty$. If the function $\phi(\zeta(x)\omega(x))$ is integrable on $[a,b]$, then
(3.16)

$$\int_a^b \Phi\left(\frac{F_1(x)\dots F_i(x)\ W_1(x)\dots W_i(x)}{x^2}\right) d_1(x),\dots,d_i(x) \le \int_a^b \Phi\left((\zeta_1(x)\dots\zeta_i(x))\ (\omega_1(x)\dots\omega_i(x))\right) d_1(x),\dots,d_i(x)$$

Proof. The result is obtained from (2.1) in connection with Chebyshev integral inequality, convexity of a functions and by considering the function $\rho(x)=1$ for all $x\in[a,b]$, see [17] we obtain.

(3.17)

$$\int_a^b \Phi\left(\frac{F_1(x)\dots F_i(x)\; W_1(x)\dots W_i(x)}{x^2}\right) d_1(x),\dots,d_i(x) =$$

$$= \int_a^b \Phi\left(\frac{1}{x^2}\left[\int_0^x (\zeta_1(t)\dots\zeta_i(t))d_1(t),\dots,d_i(t)\right]\left[\int_0^x (\omega_1(t)\dots\omega_i(t))d_1(t),\dots,d_i(t)\right] d_1(x),\dots,d_i(x)\right)$$

$$\leq \int_a^b \Phi\left(\frac{1}{x^2}\left[\int_0^x d_1(t),\dots,d_i(t)\right]\left[\int_0^x (\zeta_1(t)\dots\zeta_i(t))(\omega_1(t)\dots\omega_i(t))dt\right] d_1(x),\dots,d_i(x)\right)$$

$$= \int_a^b \Phi\left(\frac{1}{x}\left[\int_0^x (\zeta_1(t)\dots\zeta_i(t))(\omega_1(t)\dots\omega_i(t))d_1(t),\dots,d_i(t)\right] d_1(x),\dots,d_i(x)\right)$$

$$= \int_a^b \Phi\left(\frac{(\zeta_1(t)\dots\zeta_i(t))}{x}\left[\int_0^x (\omega_1(t)\dots\omega_i(t))d_1(t),\dots,d_i(t)\right] d_1(x),\dots,d_i(x)\right)$$

$$= \int_a^b \Phi\left(\frac{(\zeta_1(t)\dots\zeta_i(t))(\omega_1(t)\dots\omega_i(t))}{x}\left[\int_0^x d_1(t),\dots,d_i(t)\right] d_1(x),\dots,d_i(x)\right)$$

$$\int_a^b \Phi\left(\frac{F_1(x)\dots F_i(x)\; W_1(x)\dots W_i(x)}{x^2}\right) d_1(x),\dots,d_i(x) \leq \int_a^b \Phi\left((\zeta_1(t)\dots\zeta_i(t))(\omega_1(t)\dots\omega_i(t))\right) d_1(x),\dots,d_i(x)$$

4. Some New Refined Hardy-type Integral Inequalities

The new refined weighted Hardy-type Integral Inequalities are established as our main results:
Theorem 4.1 :
Let $r > 1, k > 1, 0 < b \leq \infty$, and let $\zeta_1(t)\dots\zeta_i(t)$ be absolutely continuous function and locally integrable on $(0, b)$ such that $0 < \int_0^b x^{r-k}\; \zeta_1^r(x)\dots\zeta_i^r(x)d_1(x),\dots,d_i(x) < \infty$.
(a) If $r \geq 1$, then
(4.1)

$$\int_0^b \frac{1}{x^k}\left(\int_0^x \zeta_1^r(t)\dots\zeta_i^r(t)d_1(t),\dots,d_i(t)\right)^r d_1(x),\dots,d_i(x)+$$

$$+\frac{k-1}{r}\int_0^b\int_t^b\left[\frac{r}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{r}}\zeta_1^r(t)\dots\zeta_i^r(t)-\frac{1}{x}\int_0^x\zeta_1^r(t)\dots\zeta_i^r(t)d_1(t),\dots,d_i(t)\right]^r x^{r-k-\frac{k-1}{r}}d_1(x),\dots,d_i(x)\; t^{\frac{k-1}{r}-1}$$

$$\leq\left(\frac{r}{k-1}\right)^r\int_0^b\left(1-\left[\frac{x}{b}\right]^{\frac{k-1}{r}}\right)^r x^{r-k}\;\zeta_1^r(x)\dots\zeta_i^r(x)d_1(x),\dots,d_i(x)$$

(b) If $0 \leq r \leq 1$, then the above inequality holds in the reserved direction.
The classical Hardy inequality (1.1) for $k > 1$ can be refined by adding a second term on the left hand side. We have the following inequality:
(4.2)

$$\int_0^\infty \frac{1}{x^k}\left(\int_0^x \zeta_1^r(t)\dots\zeta_i^r(t)d_1(t),\dots,d_i(t)\right)^2 d_1(x),\dots,d_i(x)+$$

$$+\frac{k-1}{2}\int_0^\infty\int_t^\infty\left(\frac{2}{k-1}\left[\frac{t}{x}\right]^{1-\frac{k-1}{2}}\zeta_1^r(t)\dots\zeta_i^r(t)-\frac{1}{x}\int_0^x\zeta_1^r(t)\dots\zeta_i^r(t)d_1(t),\dots,d_i(t)\right)^2\times$$

$$\times\, x^{2-k-\frac{k-1}{2}}d_1(x),\dots,d_i(x)\; t^{\frac{k-1}{2}-1}d_1(t),\dots,d_i(t)$$

$$=\left(\frac{2}{k-1}\right)^2\int_0^\infty\left(1-\left(\frac{x}{b}\right)^{\frac{k-1}{2}}\right)\; x^{2-k}\;\zeta_1^2(x)\dots\zeta_i^2(x)\; d_1(x),\dots,d_i(x)$$

Proof. . By combining Corollary 2.1, $\Phi(x) = x^r$ with $r \geq 1$ and $u(x) = 1$, yeilds
(4.3)

$$\int_0^b \left(\frac{1}{x}\int_0^x \zeta_1^r(t)\ldots\zeta_i^r(t))d_1(t),\ldots,d_i(t)\right)^r \frac{d_1(x),\ldots,d_i(x)}{x} +$$
$$+\int_0^b\int_t^b \left(\zeta_1^r(t)\ldots\zeta^r(t) - \frac{1}{x}\int_0^x \zeta_1^r(t)\ldots\zeta_i^r(t)d_1(t),\ldots,d_i(t)\right)^r \frac{d_1(x),\ldots,d_i(x)}{x^2}\, d_1(t),\ldots,d_i(t)$$
$$\leq \int_0^b \left(1-\frac{x}{b}\right)\, \zeta_1^r(x)\ldots\zeta_i^r(x)\, \frac{d_1(x),\ldots,d_i(x)}{x}$$

With $a = b^{\frac{k-1}{r}}, x = \zeta_1(t^{\frac{r}{k-1}}),\ldots,\zeta_i(t^{\frac{r}{k-1}})x^{\frac{r}{k-1}}$and $y = x^{\frac{r}{k-1}}$ and $q = t^{\frac{r}{k-1}}$. Therefore,

$$\int_0^a \left(\frac{1}{x}\int_0^x \zeta_1(t^{\frac{r}{k-1}}),\ldots,\zeta_i(t^{\frac{r}{k-1}})t^{\frac{r}{k-1}-1}d_1(t),\ldots,d_i(t)\right)^r \frac{d_1(x),\ldots,d_i(x)}{x} +$$
$$+\int_0^a\int_t^a \left(\zeta_1(t^{\frac{r}{k-1}}),\ldots,\zeta_i(t^{\frac{r}{k-1}})x^{\frac{r}{k-1}-1} - \frac{1}{x}\int_0^x \zeta_1(t^{\frac{r}{k-1}}),\ldots,\zeta_i(t^{\frac{r}{k-1}})t^{\frac{r}{k-1}-1}dt\right)^r \frac{d_1(x),\ldots,d_i(x)}{x^2} d_1(t),\ldots,d_i(t)$$
$$= \left(\frac{k-1}{r}\right)^r \int_0^a \left(\frac{1}{x}\int_0^{x^{\frac{r}{k-1}}} \zeta_1(q),\ldots,\zeta_i(q)d_1(q),\ldots,d_i(q)\right)^r \frac{d_1(x),\ldots,d_i(x)}{x} +$$
$$+\left(\frac{k-1}{r}\right)^{r+1}\int_0^b\int_{q^{\frac{k-1}{r}}}^a \left(\frac{r}{k-1}\zeta_1(q),\ldots,\zeta_i(q)q^{1-\frac{k-1}{r}} - \frac{1}{x}\int_0^{x^{\frac{r}{k-1}}} \zeta_1(q),\ldots,\zeta_i(q)dq\right)^r \times$$
$$\times \frac{d_1(x),\ldots,d_i(x)}{x^2}\; q^{\frac{k-1}{r}-1}d_1(q),\ldots,d_i(q)$$
$$\leq \int_0^a \left(1-\frac{x}{a}\right) \zeta_1^r(t^{\frac{r}{k-1}}),\ldots,\zeta_i^r(t^{\frac{r}{k-1}})x^{r(\frac{k-1}{r}-1)} \frac{d_1(x),\ldots,d_i(x)}{x}$$

which implies
(4.4)

$$\left(\frac{k-1}{r}\right)^{r+1}\int_0^b \frac{1}{y^k}\left(\int_0^y \zeta_1(q),\ldots,\zeta_i(q)d_1(q),\ldots,d_i(q)\right)^r d_1(y),\ldots,d_i(y) +$$
$$+\left(\frac{k-1}{r}\right)^{r+2}\int_0^b\int_q^b \left(\frac{r}{k-1}\zeta_1(q),\ldots,\zeta_i(q)q^{1-\frac{k-1}{r}} - \frac{1}{y^{\frac{k-1}{r}}}\int_0^y \zeta_1(q),\ldots,\zeta_i(q)d_1(q),\ldots,d_i(q)\right)^r \times$$
$$\times\, y^{\frac{1-k}{r}-1}d_1(y),\ldots,d_i(y)\; q^{\frac{k-1}{r}-1}d_1(q),\ldots,d_i(q)$$
$$= \left(\frac{k-1}{r}\right)^{r+1}\int_0^b \frac{1}{y^k}\left(\int_0^y \zeta_1(q),\ldots,\zeta_i(q)d_1(q),\ldots,d_i(q)\right)^r d_1(y),\ldots,d_i(y) +$$
$$+\left(\frac{k-1}{r}\right)^{r+2}\int_0^b\int_q^b \left(\frac{r}{k-1}\zeta_1(q),\ldots,\zeta_i(q)\left(\frac{q}{y}\right)^{1-\frac{k-1}{r}} - \frac{1}{y}\int_0^y \zeta_1(q),\ldots,\zeta_i(q)d_1(q),\ldots,d_i(q)\right)^r \times$$
$$\times\, y^{r-k\frac{1-k}{r}-1}dy\; q^{\frac{k-1}{r}-1}d_1(q),\ldots,d_i(q)$$
$$\leq \left(\frac{k-1}{r}\right)\int_0^b \left(1-\left[\frac{y}{b}\right]^{\frac{k-1}{r}}\right) y^{r-k}\zeta_1^r(y),\ldots,\zeta_i^r(y)d_1(y),\ldots,d_i(y) \quad \square$$

(b) The proof for the case $0 < p \leq 1$ is similar and the only disparity is that in the case all the inequality signs are reserved.

Remark 4.1:

Putting $u_1,\ldots,u_i = 1$ and ν the weight function is equal to

$$\begin{cases} 1-\frac{x}{b},\; b<\infty \\ 1, b\leq\infty \end{cases}$$

Thus, for $b < \infty$ and $a = 0$ (4.3) becomes

$$\int_0^b \left(\frac{1}{x}\int_0^x \zeta_1(t)\ldots\zeta_i(t)d_1(t)\ldots d_i(t)\right)^r \frac{d_1(x)\ldots d_i(x)}{x}+$$
$$+\int_0^b\int_t^b \left(\zeta_1(t)\ldots\zeta_i(t)-\frac{1}{x}\int_0^x \zeta_1(t)\ldots\zeta_i(t)d_1(t)\ldots d_i(t)\right)^r \frac{u_1(x)\ldots u_i(x)}{x^2}d_1(x)\ldots d_i(x)\ d_1(t)\ldots d_i(t)$$
$$\leq \int_0^b (\zeta_1(t)\ldots\zeta_i(t))^r\ \frac{d_1(x)\ldots d_i(x)}{x}$$ and when $b = \infty$ $a = 0$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x \zeta_1(t)\ldots\zeta_i(t)d_1(t)\ldots d_i(t)\right)^r \frac{d_1(x)\ldots d_i(x)}{x}+$$
$$+\int_0^\infty\int_t^b \left(\zeta_1(t)\ldots\zeta_i(t)-\frac{1}{x}\int_0^x \zeta_1(t)\ldots\zeta_i(t)d_1(t)\ldots d_i(t)\right)^r \frac{u_1(x)\ldots u_i(x)}{x^2}d_1(x)\ldots d_i(x)\ d_1(t)\ldots d_i(t)$$
$$\leq \int_0^\infty (\zeta_1(t)\ldots\zeta_i(t))^r\ \frac{d_1(x)\ldots d_i(x)}{x}$$

Theorem 4.2 :
Let $r > 1, k > 1, 0 < b \leq \infty$ and $\zeta_1(t)\ldots\zeta_i(t)$ be absolutely continuous function and locally integrable on (b,∞) such that $0 < \int_b^0 x^{r-k}\ \zeta_1^r(x)\ldots\zeta_i^r(x)dx_1,\ldots,dx_i < \infty$.
(c) If $r \geq 1$, then
(4.5)

$$\int_b^\infty \frac{1}{x^k}\ \left(\int_x^\infty \zeta_1(t)\ldots\zeta_i(t)dt_1,\ldots,dt_i\right)^r dx_1,\ldots,dx_i+$$
$$+\frac{k-1}{r}\int_b^\infty\int_b^t\left(\frac{r}{k-1}\left[\frac{t}{x}\right]^{1+\frac{1-k}{r}}\zeta_1^r(t)\ldots\zeta_i^r(t)-\frac{1}{x}\int_x^\infty \zeta_1(t)\ldots\zeta_i(t)dt_1,\ldots,dt_i\right)^r x^{r-k+\frac{k-1}{r}}dx_1,\ldots,dx_i.\ t^{\frac{k-1}{r}-1}dt_1,\ldots,dt_i$$
$$\leq \left(\frac{r}{k-1}\right)^r\int_b^\infty \left(1-\left[\frac{b}{x}\right]^{\frac{k-1}{r}}\right)\ x^{r-k}\ \zeta_1^r(y)\ldots\zeta_i^r(y)\ dy$$

(d) If $0 < r \leq 1$, then the above inequality holds in the reserved direction.
The classical Hardy inequality (1.1) for $k < 1$ can be refined by adding a second term on the left hand side. In particular, for $b = 0$ and $r = 2$ we have the following inequality:

(4.6)

$$\int_0^\infty \frac{1}{x^k}\ \left(\int_x^\infty \zeta_1^r(t)\ldots\zeta_i^r(t)dt_1,\ldots,dt_i\right)^2 dx_1,\ldots,dx_i+$$
$$+\frac{1-k}{2}\int_0^\infty\int_0^t\left[\frac{2}{1-k}\left(\frac{t}{x}\right)^{1+\frac{1-k}{2}}\zeta_1^r(t)\ldots\zeta_i^r(t)-\frac{1}{x}\int_x^\infty \zeta_1(t)\ldots\zeta_i(t)dt_1,\ldots,dt_i\right]^2 x^{2-k+\frac{1-k}{2}}dx_1,\ldots,dx_i.\ t^{\frac{k-1}{2}-1}dt_1,\ldots,dt_i$$
$$=\left(\frac{2}{1-k}\right)^2\int_0^\infty\ x^{2-k}\ \zeta_1^2(x)\ldots\zeta_i^2(x)\ dx_1,\ldots,dx_i$$

Proof. Corollary 2.1, $\Phi(x) = x^r$ with $r \geq 2$ and $u(x) = 1$ yields

$$\begin{aligned}&\int_0^\infty \left(x\int_x^\infty \zeta_1(t)\ldots\zeta_i(t))\frac{dt_1,\ldots,dt_i}{t^2}\right)^r \frac{dx_1,\ldots,dx_i}{x}+\\ &+\int_b^\infty\int_\infty^t \left(\zeta_1(t)\ldots\zeta_i(t)-x\int_x^\infty \zeta_1(t)\ldots\zeta_i(t)\frac{dt_1,\ldots,dt_i}{t^2}\right)^r\ dx_1,\ldots,dx_i\ \frac{dt_1,\ldots,dt_i}{t^2}\\ &\leq \int_0^b\left(1-\frac{x}{b}\right)\ \zeta_1^r(x)\ldots\zeta_i^r(x)\ \frac{dx_1,\ldots,dx_i}{x}\end{aligned} \tag{4.7}$$

By replacing $a = b^{\frac{1-k}{r}}, x = \zeta_1(t^{\frac{r}{1-k}}), \dots, \zeta_i(t^{\frac{r}{1-k}})x^{\frac{r}{1-k}}$. Thereafter, by using the substitution $y = x^{\frac{r}{1-k}}$ and $q = t^{\frac{r}{1-k}}$ implies

$$\int_b^\infty \left(x \int_x^\infty \zeta_1(t^{\frac{r}{1-k}}), \dots, \zeta_i(t^{\frac{r}{1-k}}) t^{\frac{r}{1-k}-1} \frac{dt_1, \dots, dt_i}{t^2} \right)^r \frac{dx_1, \dots, dx_i}{x} +$$
$$+ \int_a^\infty \int_a^t \left(\zeta_1(t^{\frac{r}{1-k}}), \dots, \zeta_i(t^{\frac{r}{k-1}}) x^{\frac{r}{1-k}-1} - x \int_x^\infty \zeta_1(t^{\frac{r}{1-k}}), \dots, \zeta_i(t^{\frac{r}{1-k}}) t^{\frac{r}{1-k}-1} dt \right)^r dx_1, \dots, dx_i \, \frac{dt_1, \dots, dt_i}{t^2}$$
$$= \left(\frac{1-k}{r} \right)^r \int_a^\infty \left(x \int_{x^{\frac{r}{1-k}}}^\infty \zeta_1(q), \dots, \zeta_i(q) dq \right)^r \frac{dx_1, \dots, dx_i}{x} +$$
$$+ \left(\frac{1-k}{r} \right)^{r+1} \int_b^\infty \int_a^{q^{\frac{1-k}{r}}} \left(\frac{r}{1-k} \zeta_1(q), \dots, \zeta_i(q) q^{1-\frac{1-k}{r}} - x \int_{x^{\frac{r}{1-k}}}^\infty \zeta_1(q), \dots, \zeta_i(q) dq_1, \dots, dq_i \right)^r dx_1, \dots, dx_i \, q^{\frac{k-1}{r}-1} dq_1, \dots, dq_i$$
$$\leq \int_0^\infty \left(1 - \frac{a}{x} \right) \zeta_1^r(t^{\frac{r}{1-k}}), \dots, \zeta_i^r(t^{\frac{r}{1-k}}) x^{r(\frac{r}{1-k}+1)} \frac{dx_1, \dots, dx_i}{x}$$

that is

(4.8)

$$\left(\frac{1-k}{r} \right)^{r+1} \int_b^\infty \frac{1}{y^k} \left(\int_y^\infty \zeta_1(q), \dots, \zeta_i(q) dq_1, \dots, dq_i \right)^r dy_1, \dots, dy_i +$$
$$+ \left(\frac{k-1}{r} \right)^{r+2} \int_b^\infty \int_b^q \left(\frac{r}{k-1} \zeta_1(q), \dots, \zeta_i(q) q^{1-\frac{1-k}{r}} - \frac{1}{y^{\frac{k-1}{r}}} \int_0^y \zeta_1(q), \dots, \zeta_i(q) dq \right)^r \times$$
$$\times y^{\frac{1-k}{r}-1} dy_1, \dots, dy_i \, q^{\frac{k-1}{r}-1} dq_1, \dots, dq_i$$
$$= \left(\frac{k-1}{r} \right)^{r+1} \int_0^\infty \frac{1}{y^k} \left(\int_0^y \zeta_1(q), \dots, \zeta_i(q) dq_1, \dots, dq_i \right)^r dy_1, \dots, dy_i +$$
$$+ \left(\frac{k-1}{r} \right)^{r+2} \int_0^\infty \int_q^b \left(\frac{r}{k-1} \zeta_1(q), \dots, \zeta_i(q) q^{1+\frac{1-k}{r}} - y^{1+\frac{1-k}{r}} \int_y^\infty \zeta_1(q), \dots, \zeta_i(q) dq_1, \dots, dq_i \right)^r \times$$
$$\times y^{\frac{1-k}{r}-1} dy_1, \dots, dy_i \, q^{\frac{k-1}{r}-1} dq_1, \dots, dq_i$$
$$= \left(\frac{k-1}{r} \right)^{r+1} \int_0^\infty \frac{1}{y^k} \left(\int_0^y \zeta_1(q), \dots, \zeta_i(q) dq_1, \dots, dq_i \right)^r dy_1, \dots, dy_i +$$
$$+ \left(\frac{k-1}{r} \right)^{r+2} \int_b^\infty \int_b^q \left(\frac{r}{k-1} \zeta_1(q), \dots, \zeta_i(q) \left[\frac{q}{y} \right]^{1+\frac{1-k}{r}} - \frac{1}{y} \int_y^\infty \zeta_1(q), \dots, \zeta_i(q) dq_1, \dots, dq_i \right)^r \times$$
$$\times y^{\frac{1-k}{r} q - k} dy_1, \dots, dy_i \, q^{\frac{k-1}{r}-1} dq_1, \dots, dq_i$$
$$\leq \left(\frac{1-k}{r} \right) \int_b^\infty \left(1 - \left[\frac{b}{y} \right]^{\frac{1-k}{r}} \right) y^{r-k} \zeta_1^r(y), \dots, \zeta_i^r(y) dy_1, \dots, dy_i \quad \square$$

This completes the proof of the Theorem.

(d) The inequalities are reversed in the case of $0 < p \leq 1$ and the proof is similar.

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