CONVECTION–DIFFUSION EQUATION IN UNIFORMLY LOCAL LEBESGUE SPACES

ABSTRACT. In this paper we establish the local existence and uniqueness of the mild solution to the Cauchy problem for convection–diffusion equation in n-dimensional Euclidean space with initial data in uniformly local function spaces $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. For the proof, we utiliize the uniformly local $L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$- $L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimate for the convolution operators got by Maekawa and Terasawa [22], and the Banach fixed point hypothesis.

1. INTRODUCTION

In $\mathbb{R}^n$, we take into account the Cauchy problem of the convection–diffusion equation. For $m > 1$ and $b \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{cases} \partial_t v - \Delta v = b \cdot \nabla(|v|^{m-1}u), & t > 0,\ x \in \mathbb{R}^n, \\ \quad v(0,x) = v_0(x), & \qquad\quad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $v_0 = v_0(x)$; $\mathbb{R}^n \to \mathbb{R}$ is the initial data and $v = v(t,x)$; $\mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is an unknown function. Our fundamental reason here is to solve (1.1) and establish the well-posedness for initial condition which may not decay at space infinity but rather not be locally bounded.

For any initial condition $v_0 \in L^1(\mathbb{R}^n)$, Escobedo and Zuazua [8] proved that there exists a unique global classical solution $v \in C([0,\infty); L^1(\mathbb{R}^n))$ of (1.1) with

$$v \in C((0,\infty); W^{2,s}(\mathbb{R}^n)) \cap C^1((0,\infty); L^s(\mathbb{R}^n)), \tag{1.2}$$

for any $s \in (1,\infty)$. In addition, they demonstrated decay properties when the initial condition is in $L^1(\mathbb{R}^n)$ and established the large–time behavior of solutions to (1.1).

Numerous authors have considered problem (1.1) (see, e.g., [1], [6], [7], [8], [9], [10], [11], [13], [14], [21], [24], [27]).

In contrast, in [2], [3], [5], [12], [19], [20], [22] and [23], the authors employ spaces of functions whose elements have a uniform size when measured in balls of arbitrary center but fixed radius. Uniformly local spaces are the name given to these areas. For solving parabolic equations in unbounded domains with non-decaying initial functions, these spaces are natural and useful. In addition to belonging to any constant functions, the

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spaces have compact embeddings and appropriate inclusion properties. Particularly, the definition of uniformly local Lebesgue space is straightforward, and it is evident that it contains some functions that may have singularities and may not decay at infinity of space. Additionally, when time reaches zero, mild solution convergences to initial data are relatively straightforward. Maekawa and Terasawa [22] established the $L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimates for convolution kernels including $e^{t\Delta}$, $\nabla e^{t\Delta}$ and $e^{t\Delta}\mathbf{P}\nabla$ by constructing a mild solution of the Navier-Stokes equations with initial condition in uniformly local Lebesgue spaces. Haque–Ogawa–Sato [16] demonstrated the existence and uniqueness of weak solutions to (1.1) with initial data in uniformly local function spaces $\mathcal{L}^r_{\mathrm{uloc},\rho}(\Omega)$. In order to accomplish this, they presented the semigroup method solution for $BUC(\Omega)$, bounded uniformly continuous functions. In this paper, we broaden the result included in [16] into uniformly local Lebesgue spaces.

Definition (Uniformly local Lebesgue spaces). Let $1 \le r \le \infty$ and $\rho > 0$. The Lebesgue spaces on $\mathbb{R}^n$ that are uniformly local and are denoted by $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, are defined as

$$L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n) := \left\{ g \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \ \|g\|_{L^r_{\mathrm{uloc},\rho}} < \infty \right\},$$

where for $\rho > 0$

$$\|g\|_{L^r_{\mathrm{uloc},\rho}} = \begin{cases} \sup\limits_{x\in\mathbb{R}^n} \left(\displaystyle\int_{B_\rho(x)} |g(y)|^r dy \right)^{\frac{1}{r}}, & 1 \le r < \infty, \\ \sup\limits_{x\in\mathbb{R}^n} \sup\limits_{y\in B_\rho(x)} |g(y)|, & r = \infty. \end{cases} \tag{1.3}$$

We distinguish here $L^\infty_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ as $L^\infty(\mathbb{R}^n)$. $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ which is a uniformly local Lebesgue spaces is a Banach space with the standard characterized in (1.3). We characterize the subspace $\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ as the closure of the space of continuous functions which is uniformly bounded, $BUC(\mathbb{R}^n)$ in the space $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, i.e.,

$$\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n) := \overline{BUC(\mathbb{R}^n)}^{\|\cdot\|_{L^r_{\mathrm{uloc},\rho}}}$$

and the definition of $\mathcal{L}^\infty_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ is $\mathcal{L}^\infty_{\mathrm{uloc},\rho}(\mathbb{R}^n) = BUC(\mathbb{R}^n)$.

We transform the equation into the following form of integral equation in order to solve (1.1)

$$v(t) = e^{t\Delta}v_0 + \int_0^t b \cdot \nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)ds. \tag{1.4}$$

Here $e^{t\Delta}v_0$ is distinguish the heat semigroup. The solution which we obtain from the integral equation (1.4) is referred to as the mild solution of (1.1) with initial condition u_0. The exact importance of each term and the way that they are well defined follows from the uniformly local $L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimates of convolution type operators acquired in [3] and [22].

We express our principal result for the existence of a mild solution for (1.1) in uniformly local Lebesgue spaces.

Theorem 1.1 (Existence and uniqueness)**.** *Assume $m > 1$ and $1 \le r < \infty$ with the conditions*

$$\begin{cases} n(m-1) < r & \text{if} \quad 1 + \frac{1}{n} < m, \\ 1 < r & \text{if} \quad 1 + \frac{1}{n} = m, \\ 1 \le r & \text{if} \quad 1 + \frac{1}{n} > m > 1. \end{cases} \tag{1.5}$$

Then, for every $v_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, there is a $T > 0$, depens on n, m, r and $\|v_0\|_{L^r_{\mathrm{uloc},\rho}}$ and a unique mild solution $v \in L^\infty(0,T; \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)) \cap C((0,T); \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$ of (1.1).

Theorem 1.2 (Convergence to initial data)**.** *Assume the same circumstances as in Theorem 1.1. Consider $v \in L^\infty(0,T; \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$ be a mild unique solution with initial condition $v_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. Then we have got*

$$\|v(t) - v_0\|_{L^r_{\mathrm{uloc},\rho}} \to 0 \quad as \quad t \to 0. \tag{1.6}$$

This paper is coordinated as follows. We will list some uniformly local space properties in Section 2. We will present estimates of the $L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ type for convolution operators with integrable functions that satisfy a few conditions in Section 3. By applying these estimates, we will develope mild solutions of (1.1). In section 4, we will demonstrate our principal Theorem 1.1 and Theorem 1.2 utilizing the estimates that expressed in segment 3.

2. Uniformly local spaces and their properties

In this segment, we express a few properties of the function class $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ and $\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. We use the abbreviations like $L^r_{\mathrm{uloc}}(\mathbb{R}^n) := L^r_{\mathrm{uloc},1}(\mathbb{R}^n)$ and $\mathcal{L}^r_{\mathrm{uloc}}(\mathbb{R}^n) := \mathcal{L}^r_{\mathrm{uloc},1}(\mathbb{R}^n)$ when $\rho = 1$, . The following are the inclusion relations for $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ spaces:

Proposition 2.1. (1) *For every $\rho_1, \rho_2 > 0$, we know $L^r_{\mathrm{uloc},\rho_1}(\mathbb{R}^n) = L^r_{\mathrm{uloc},\rho_2}(\mathbb{R}^n)$ with the norm of equivalence.*
(2) *For any $1 \le p \le q \le \infty$ and $\rho > 0$, we have $L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n) \subset L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.*
(3) *Let $1 \le r < \infty$ and $\rho > 0$, we have $L^r(\mathbb{R}^n) \subset L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, $L^\infty(\mathbb{R}^n) \subset L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.*

Proof of Proposition 2.1. The inclusions (1), (2) and (3) of the Proposition 2.1 follow from the Hölder inequality, we omit the detail. ❑

Proposition 2.2. *The class of compact supported smooth functions; $C_0^\infty(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ are not dense in $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.*

Proof of Proposition 2.2. We know that $1 \in L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. Let $g \in C_0^\infty(\mathbb{R}^n)$. We may assume that there exist $R_f > 0$ such that supp $g \subset B_{R_f}(0)$. Then

$$\begin{aligned}\|g-1\|_{L^r_{\mathrm{uloc},\rho}} &= \sup_{x\in\mathbb{R}^n}\left(\int_{B_\rho(x)} |1-g(y)|^r dy\right)^{\frac{1}{r}}\\ &\geq \left(\int_{B_\rho(x))} |1-g(y)|^r dy\right)^{\frac{1}{r}}.\end{aligned}$$

Take $x_0 \in \mathbb{R}^n$ such that $|x_0| > R_g + \rho$. Then $g(y) = 0$ on $B_\rho(x_0)$.
Now

$$\begin{aligned}\|g-1\|_{L^r_{\mathrm{uloc},\rho}} &\geq \left(\int_{B_\rho(x_0)} 1dy\right)^{\frac{1}{r}}\\ &=|B_\rho(x_0)|^{\frac{1}{r}} = |B_\rho(0)|^{\frac{1}{r}}.\end{aligned}$$

This holds for any $g \in C_0^\infty(\mathbb{R}^n)$. Hence for all $g \in C_0^\infty(\mathbb{R}^n)$ can not approximates $1 \in L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. This stands that $C_0^\infty(\mathbb{R}^n)$ is not dense in $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.

Assume $f \in C_0^\infty(\mathbb{R}^n)$ be a function satisfies that *supp* $f \subset B_1(0)$, $\int_{\mathbb{R}^n} |f|^r dx = 1$ and $f \geq 0$. Choose any countable points $\{x_m\}_m \geq 1$ such that $B_1(x_m) \cap B_1(x_{m'}) = \emptyset$. Set $f_m(x) = m^{\frac{n}{r}} f(m(x-x_m))$. Then *supp* $f_m \subset B_{\frac{1}{m}}(x_m)$. So if we set

$$\tilde{f}(x) = \begin{cases} f_m(x), & \text{for } \ x \in B_1(x_m),\\ 0, & \text{otherwise,}\end{cases}$$

then we have

$$\|\tilde{f}\|_{L^r_{\mathrm{uloc}}(\mathbb{R}^n)} = \sup_m \|f_m\|_{L^r(B_1(x_m))} = 1.$$

Suppose $q > r$. For every $g \in L^r_{\mathrm{uloc}}(\mathbb{R}^n)$, we know that $\|\tilde{f} - g\| L^r_{\mathrm{uloc}}(\mathbb{R}^n) \geq 1$. To be sure, we have

$$\begin{aligned}\|\tilde{f}-g\|_{L^r_{\mathrm{uloc}}(\mathbb{R}^n)} &\geq \|f_m - g\|_{L^r(B_1(x_m))} \geq \|f_m - g\|_{L^r(B_{\frac{1}{m}}(x_m))}\\ &\geq \left|\|f_m\|_{L^r(B_{\frac{1}{m}}(x_m))} - \|g\|_{L^r(B_{\frac{1}{m}}(x_m))}\right| \geq 1 - |B_{\frac{1}{m}}(x_m)|^{\frac{1}{r}(1-\frac{r}{q})}\|g\|_{L^q_{\mathrm{uloc}}(\mathbb{R}^n)}\\ &\geq 1 - \frac{C}{m^n(\frac{1}{r}-\frac{1}{q})}\|g\|_{L^q_{\mathrm{uloc}}(\mathbb{R}^n)} \to 1 \quad \text{as} \quad m \to \infty.\end{aligned}$$

This implies that $\tilde{f}$ does not belong to $\overline{\cup_{q>r} L^q_{\mathrm{uloc}}(\mathbb{R}^n)}^{L^r_{\mathrm{uloc}}(\mathbb{R}^n)}$. As a result, we know that the subset $\cup_{q>r} L^q_{\mathrm{uloc}}(\mathbb{R}^n)$ does not have a dense in $L^r_{\mathrm{uloc}}(\mathbb{R}^n)$ and implies that $L^\infty(\mathbb{R}^n)$ is not dense in $L^r_{\mathrm{uloc}}(\mathbb{R}^n)$. Hence the space $L^\infty(\mathbb{R}^n)$ not dense in $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. ❑

We have the following characterizations of $\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. These characterizations are obtained in [3] and [22].

Proposition 2.3. *For every positive ρ, the accompanying three properties are equivalent:*
(1) $g \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.

(2) $\lim_{|s|\to 0} \|g(\cdot+s)-g(\cdot)\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)}=0$.
(3) $\lim_{t\to 0+} \|e^{t\Delta}g-g\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)}=0$.

For the proof see [22].

3. Mild solutions in uniformly local spaces

In this part, we will give the meaning of mild solutions (solutions of the integral equations) for the convection–diffusions equation (1.1). Key estimates for the convolution operators $e^{t\Delta}$ and $\nabla e^{t\Delta}$, which are derived in [3] and [22] will also discussed.

Definition (Mild solution). Assume the conditions $m>1$ and $T>0$. The function $v\in L^\infty(0,T;\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$ is known as a mild solution of the convection–diffusion equations (1.1) on $(0,T)\times\mathbb{R}^n$ if there is a $v_0\in\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ that satisfies the integral equation

$$v(t)=e^{t\Delta}v_0+\int_0^t b\cdot\nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)ds \tag{3.1}$$

in $C((0,T);\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$.

In [22], the following theorem is demonstrated.

Theorem 3.1. *Assume $1\le q\le p\le\infty$ and suppose that G_t is the heat kernel in $\mathbb{R}^n$. We set $G_t(x)=t^{-\frac{n}{2}}G_1(\frac{x}{\sqrt{t}})$ for $t>0$. Then, for every function $f\in L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, we can characterize pointwise*

$$G_t*f(x)=\int_{\mathbb{R}^n}G_t(x-y)f(y)dy.$$

Furthermore, we have the estimate

$$\|G_t*f\|_{L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\le\left(\frac{C_1\|G_1\|_{L^1(\mathbb{R}^n)}}{\rho^{n(\frac{1}{q}-\frac{1}{p})}}+\frac{C_2\|G_1\|_{L^r(\mathbb{R}^n)}}{t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}}\right)\|f\|_{L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)}, \tag{3.2}$$

where r satisfies the condition $\frac{1}{p}=\frac{1}{r}+\frac{1}{q}-1$, as well as C_1, C_2 are constants which are positives depends only on n.

For the verification see [22].

Estimates of $L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ for the linear and nonlinear terms in the integral equation (3.1) can be derived from Theorem 3.1.

Corollary 3.2 ($L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimates)**.** *Assuming $1\le q\le p\le\infty$. Then for every $f\in L^p_{uloc,\rho}(\mathbb{R}^n)$, we know that*

$$\|e^{t\Delta}f\|_{L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\le\left(\frac{C_1}{\rho^{n(\frac{1}{q}-\frac{1}{p})}}+\frac{C_2}{t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}}\right)\|f\|_{L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)}, \tag{3.3}$$

$$\|\nabla e^{t\Delta}f\|_{L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\le\left(\frac{C_3}{t^{\frac{1}{2}}\rho^{n(\frac{1}{q}-\frac{1}{p})}}+\frac{C_4}{t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}}}\right)\|f\|_{L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)}, \tag{3.4}$$

satisfies. Here the positive constants C_1 and C_3 depend only on n as well as the positive constants C_2 and C_4 depend only on n, p and q.

For the verification see [22].

Remark: The heat semigroup $L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimate is also obtained in [3]. The estimate is found in [15] when $p = q = \infty$.

4. Proof of Theorem

Proof of Theorem 1.1. For any positive M and any positive T. Assume $1 \le r < \infty$, and we set

$$Y = L^\infty(0,T; L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)) \cap L^\infty_{loc}(0,T; L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n))$$

and

$$Y_{M,T,r} := \left\{ v \in Y : \sup_{0<t<T} \|u(t)\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \le M \quad \text{and} \quad \sup_{0<t<T} t^{\frac{\alpha}{2}} \|u(t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \le M \right\},$$

where $\alpha = \frac{n(m-1)}{mr} < \frac{1}{m} < 1$ and M and T are constants depending on $\|v_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)}$, r, m and n, determined later. For every v and $w \in Y_{M,T,r}$, we define

$$d_{Y_{M,T,r}}(v,w) = \|v - w\|_{Y_{M,T,r}} \equiv \sup_{0<t<T} t^{\frac{\alpha}{2}} \|v(t) - u(t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}.$$

. This implies that $(Y_{M,T,r}, d_{Y_{M,T,r}})$ is a complete metric space. Then we consider a map

$$\Phi : Y_{M,T,r} \to Y_{M,T,r} \qquad \text{by}$$

$$\Phi[v](t) = e^{t\Delta} v_0 + \int_0^t b \cdot \nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1} v(s)\big) ds, \qquad v_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n), \tag{4.1}$$

and want to that Φ is a contraction mapping from $Y_{M,T,r}$ onto itself. This implies an existence of the fixed point for the map Φ on $Y_{M,T,r}$ and it becomes a solution to the coresponding integral equation (3.1) has a unique fixed point and it becomes an $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-mild solution.

For any $v \in X_{M,T,r}$, we will estimate $\|\Phi[v](t)\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)}$ and $t^{\frac{\alpha}{2}}\|\Phi[v](t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}$. By taking $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimate, we obtain

$$\begin{aligned} \|e^{t\Delta} v_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} &\le \left(\frac{C_1}{\rho^{n(\frac{1}{r} - \frac{1}{r})}} + \frac{C_2}{t^{\frac{n}{2}(\frac{1}{r} - \frac{1}{r})}} \right) \|v_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &= C \|v_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)}, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\big\|\nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1} v(s)\big)\big\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\qquad \le \left(\frac{C_3}{(t-s)^{\frac{1}{2}} \rho^{n(\frac{1}{r} - \frac{1}{r})}} + \frac{C_4}{(t-s)^{\frac{n}{2}(\frac{1}{r} - \frac{1}{r}) + \frac{1}{2}}} \right) \||v(s)|^{m-1} v(s)\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\qquad = C'(t-s)^{-\frac{1}{2}} \|v(s)\|^m_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}. \end{aligned} \tag{4.3}$$

Then, we obtain

$$
\begin{aligned}
|b| \int_0^t & \big\| \nabla e^{(t-s)\Delta} \big(|v(s)|^{m-1} v(s) \big) \big\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} ds \\
& \le C' \int_0^t (t-s)^{-\frac{1}{2}} \|v(s)\|^m_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} ds \\
& \le C' \big(\sup_{0<t<T} t^{\frac{\alpha}{2}} \|v(t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \big)^m \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{m\alpha}{2}} ds \\
& \le C' M^m t^{\frac{1}{2}-\frac{m\alpha}{2}} \beta\big(\frac{1}{2}, 1 - \frac{m\alpha}{2}\big).
\end{aligned} \tag{4.4}
$$

The integral and $t^{\frac{1}{2}-\frac{m\alpha}{2}}$ is bounded if and only if $1-\frac{m\alpha}{2} > 0$ and $\frac{1}{2}-\frac{m\alpha}{2} \ge 0$, that is, $r > n(m-1)$. Thus if these conditions are satisfied, we have

$$
\|\Phi[v](t)\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \le C\|v_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} + C' M^m T^{\frac{1}{2}-\frac{m\alpha}{2}} \beta\big(\frac{1}{2}, 1 - \frac{m\alpha}{2} \big). \tag{4.5}
$$

Next we estimate $t^{\frac{\alpha}{2}} \|\Phi[v](t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}$. By taking $L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimate and choose $\rho \ge t^{\frac{1}{2}}$, we obtain

$$
\begin{aligned}
t^{\frac{\alpha}{2}} \|e^{t\Delta} v_0\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} & \le t^{\frac{\alpha}{2}} \left(\frac{C_1}{\rho^{n(\frac{1}{r}-\frac{1}{mr})}} + \frac{C_2}{t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{mr})}} \right) \|v_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
& \le t^{\frac{\alpha}{2}} t^{-\frac{\alpha}{2}} \|v_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} = C \|v_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)},
\end{aligned} \tag{4.6}
$$

and

$$
\begin{aligned}
\big\| \nabla & e^{(t-s)\Delta} \big(|v(s)|^{m-1} v(s) \big) \big\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
& \le \left(\frac{C_3}{(t-s)^{\frac{1}{2}} \rho^{n(\frac{1}{r}-\frac{1}{mr})}} + \frac{C_4}{(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{mr})+\frac{1}{2}}} \right) \| |v(s)|^{m-1} v(s) \|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
& = C' (t-s)^{-\frac{1}{2}-\frac{\alpha}{2}} \|v(s)\|^m_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}.
\end{aligned} \tag{4.7}
$$

Then, we obtain

$$
\begin{aligned}
|b|\ t^{\frac{\alpha}{2}} \int_0^t & \big\| \nabla e^{(t-s)\Delta} \big(|v(s)|^{m-1} v(s) \big) \big\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} ds \\
& \le C' t^{\frac{\alpha}{2}} \int_0^t (t-s)^{-\frac{1}{2}-\frac{\alpha}{2}} \|v(s)\|^m_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} ds \\
& \le C' t^{\frac{\alpha}{2}} \big(\sup_{0<t<T} t^{\frac{\alpha}{2}} \|v(t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \big)^m \int_0^t (t-s)^{-\frac{1}{2}-\frac{\alpha}{2}} s^{-\frac{m\alpha}{2}} ds \\
& \le C' M^m t^{\frac{1}{2}-\frac{m\alpha}{2}} \beta\big(\frac{1}{2} - \frac{\alpha}{2}, 1 - \frac{m\alpha}{2} \big).
\end{aligned} \tag{4.8}
$$

The integral and $t^{\frac{1}{2}-\frac{m\alpha}{2}}$ is bounded if and only if $\frac{1}{2}-\frac{\alpha}{2} > 0$, $1-\frac{m\alpha}{2} > 0$ and $\frac{1}{2}-\frac{m\alpha}{2} \ge 0$, that is, $\alpha < 1$ and $r > n(m-1)$. Thus if these conditions are satisfied, we have

$$
t^{\frac{\alpha}{2}} \|\Phi[v](t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \le C\|v_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} + C' M^m T^{\frac{1}{2}-\frac{m\alpha}{2}} \beta\big(\frac{1}{2} - \frac{\alpha}{2}, 1 - \frac{m\alpha}{2} \big). \tag{4.9}
$$

We next consider the condition in which Φ is contraction. Let $v_0,\, w_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. We have for any $v,\, w \in Y_{M,T,r}$ and $t \in (0,T)$

$$
\begin{aligned}
t^{\frac{\alpha}{2}}\|\Phi[v](t)-\Phi[w](t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} &\le t^{\frac{\alpha}{2}}\|e^{t\Delta}(v_0-w_0)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
&\quad + |a| t^{\frac{\alpha}{2}} \int_0^t \big\|\nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1}v(s)-|w(s)|^{m-1}w(s)\big)\big\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} ds.
\end{aligned} \tag{4.10}
$$

By taking $L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimate and choose $\rho \ge t^{\frac{1}{2}}$, we obtain

$$
\begin{aligned}
t^{\frac{\alpha}{2}}\|e^{t\Delta}v_0 - e^{t\Delta}w_0\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} &\le t^{\frac{\alpha}{2}}\left(\frac{C_1}{\rho^{n(\frac{1}{r}-\frac{1}{mr})}} + \frac{C_2}{t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{mr})}}\right)\|v_0-w_0\|_{L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
&\le C t^{\frac{\alpha}{2}} t^{-\frac{\alpha}{2}}\|v_0-w_0\|_{L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
&= C\|v_0-w_0\|_{L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}.
\end{aligned} \tag{4.11}
$$

Again, by taking $L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimate, choose $\rho \ge (t-s)^{\frac{1}{2}}$ and then applying the Hölder inequality, we obtain

$$
\begin{aligned}
&\big\|\nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1}v(s)-|w(s)|^{m-1}w(s)\big)\big\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
&\quad\le \left(\frac{C_3}{(t-s)^{\frac{1}{2}}\rho^{n(\frac{1}{r}-\frac{1}{mr})}} + \frac{C_4}{(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{mr})+\frac{1}{2}}}\right)\Big\||v(s)|^{m-1}v(s)-|w(s)|^{m-1}w(s)\Big\|_{L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
&\quad\le \left(\frac{C_3}{(t-s)^{\frac{1}{2}}\rho^{n(\frac{1}{r}-\frac{1}{mr})}} + \frac{C_4}{(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{mr})+\frac{1}{2}}}\right) \\
&\qquad\qquad \Big\|\big(m\max\{|v(s)|^{m-1},|w(s)|^{m-1}\}|v(s)-w(s)|\big)\Big\|_{L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
&\quad\le \left(\frac{C_3}{(t-s)^{\frac{1}{2}}(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{mr})}} + \frac{C_4}{(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{mr})+\frac{1}{2}}}\right) \\
&\qquad\qquad \Big\|\big(m\max\{|v(s)|^{m-1},|w(s)|^{m-1}\}|v(s)-w(s)|\big)\Big\|_{L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
&\quad\le C'(t-s)^{-\frac{1}{2}-\frac{\alpha}{2}}\|v(s)-w(s)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\big\|\max\{|v(s)|^{m-1},|w(s)|^{m-1}\}\big\|_{L^{\frac{mr}{m-1}}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
&\quad\le C'(t-s)^{-\frac{1}{2}-\frac{\alpha}{2}}\|v(s)-w(s)\|_{L^{mr}_{\mathrm{uloc},\rho}}\big(\|\,|v(s)|^{m-1}\|_{L^{\frac{mr}{m-1}}_{\mathrm{uloc},\rho}} + \|\,|w(s)|^{m-1}\|_{L^{\frac{mr}{m-1}}_{\mathrm{uloc},\rho}}\big) \\
&\quad= C'(t-s)^{-\frac{1}{2}-\frac{\alpha}{2}}\|v(s)-w(s)\|_{L^{mr}_{\mathrm{uloc},\rho}}\big(\|v(s)\|^{m-1}_{L^{mr}_{\mathrm{uloc},\rho}} + \|w(s)\|^{m-1}_{L^{mr}_{\mathrm{uloc},\rho}}\big).
\end{aligned} \tag{4.12}
$$

Hence we have

$$\sup_{t\in(0,T)} t^{\frac{\alpha}{2}}\|\Phi[v](t)-\Phi[w](t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}$$
$$\le C\|v_0-w_0\|_{L^r_{uloc,\rho}} + C'M^{m-1}\sup_{t\in(0,T)} t^{\frac{\alpha}{2}}\|v(t)-w(t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\int_0^t (t-s)^{-\frac{1}{2}-\frac{\alpha}{2}}s^{-\frac{\alpha}{2}(m-1)}ds$$
$$\le C\|v_0-w_0\|_{L^r_{uloc,\rho}} + C'M^{m-1}t^{\frac{1}{2}-\frac{\alpha m}{2}}\sup_{t\in(0,T)} t^{\frac{\alpha}{2}}\|v(t)-w(t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\beta(\frac{1}{2}-\frac{\alpha}{2},1-\frac{\alpha(m-1)}{2}). \tag{4.13}$$

The integral and $t^{\frac{1}{2}-\frac{m\alpha}{2}}$ is bounded if and only if $\frac{1}{2}-\frac{\alpha}{2}>0$, $1-\frac{(m-1)\alpha}{2}>0$ and $\frac{1}{2}-\frac{m\alpha}{2}\ge 0$, that is, $\alpha<1$ and $r>n(m-1)$. Thus if these conditions are satisfied, we have

$$\sup_{t\in(0,T)} t^{\frac{\alpha}{2}}\|\Phi[v](t)-\Phi[w](t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}$$
$$\le C\|v_0-w_0\|_{L^r_{uloc,\rho}} + C'M^{m-1}t^{\frac{1}{2}-\frac{\alpha m}{2}}\sup_{t\in(0,T)} t^{\frac{\alpha}{2}}\|v(t)-w(t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}$$
$$\le C\|v_0-w_0\|_{L^r_{uloc,\rho}} + C'M^{m-1}T^{\frac{1}{2}-\frac{\alpha m}{2}}\sup_{t\in(0,T)} t^{\frac{\alpha}{2}}\|v(t)-w(t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}. \tag{4.14}$$

Setting $v_0=w_0$, we obtain

$$d_{Y_{M,T,r}}(\Phi[v],\Phi[w])\le CM^{m-1}T^{\frac{1}{2}-\frac{\alpha m}{2}}d_{Y_{M,T,r}}(v,w).$$

It follows from the above estimates that if T is small enough then the map Φ is contraction from $Y_{M,T,r}$ onto $Y_{M,T,r}$ and by virtue of the Banach fixed point principle, there is a unique fixed point of Φ in $Y_{M,T,r}$. By the definition, this fixed point satisfies the integral equation (3.1) and besides, $v(t)\to v_0$ as $t\to 0$ by Theorem 1.2. Hence v is the $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$- mild solution to (1.1). This shows the existence of solution.

The uniqueness of the mild solution in $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ is then demonstrated. With the initial data $v_0\in\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, suppose v_1, $v_2\in L^\infty(0,T;\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$ be two mild solutions. Then by (4.14), we can obtain for any $0<t<T'<T$

$$\sup_{t\in(0,T')} t^{\frac{\alpha}{2}}\|v_1(t)-v_2(t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}$$
$$\le CM^{m-1}T'^{\frac{1}{2}-\frac{\alpha m}{2}}\sup_{t\in(0,T')} t^{\frac{\alpha}{2}}\|v_1(t)-v_2(t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}.$$

Therefore, it follows that $v_1=v_2$ in $0<t<T'$ for sufficently small $T'>0$. Repetition of this argument, we see that $v_1=v_2$ in $0<t<T$.

We have to prove taht $v\in C((0,T);\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$. Let $0<t<t+h<T$. Since

$$v(t)=e^{t\Delta}v_0+\int_0^t b\cdot\nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)ds,$$

we have

$$
\begin{aligned}
&\|v(t+h)-v(t)\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\\
\le&\|e^{(t+h)\Delta}v_0-e^{t\Delta}v_0\|_{L^{mr}_{uloc,\rho}}\\
+&\Big\|\int_0^{t+h} b\cdot\nabla e^{(t+h-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)ds-\int_0^t b\cdot\nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)ds\Big\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}.
\end{aligned}
\tag{4.15}
$$

Applying Corollary 3.2 we have $e^{t\Delta}v_0\in\mathcal{L}^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ for every positive t. As a result , by Proposition 2.3, we conclude that

$$
\|e^{(t+h)\Delta}v_0-e^{t\Delta}v_0\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}=\|e^{h\Delta}e^{t\Delta}v_0-e^{t\Delta}v_0\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\to 0 \text{ as } h\to 0.
$$

Next we have ,

$$
\begin{aligned}
&\Big\|\int_0^{t+h} b\cdot\nabla e^{(t+h-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)ds-\int_0^t b\cdot\nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)ds\Big\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\\
&\le|b|\int_0^t\big\|\nabla e^{(t+h-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)-\nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)\big\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}ds\\
&\qquad+|b|\int_t^{t+h}\big\|\nabla e^{(t+h-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)\big\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}ds=I_1+I_2.
\end{aligned}
\tag{4.16}
$$

Again, from Corollary 3.2 we know that $\nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)\in\mathcal{L}^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ for $0\le s<t$, therefore, the Proposition 2.3 implies that

$$
\big\|\nabla e^{(t+h-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)-\nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)\big\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\to 0 \text{ as } h\to 0,
$$

for any $t>s\ge 0$.

Then again, by taking $L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimate, we acquire

$$
\begin{aligned}
&\big\|\nabla e^{(t+h-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)-\nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)\big\|_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\\
\le&\bigg(\frac{C_3}{(t+h-s)^{\frac12}\rho^{n(\frac1r-\frac1{mr})}}+\frac{C_4}{(t+h-s)^{\frac n2(\frac1r-\frac1{mr})+\frac12}}\bigg)\|v(s)\|^m_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\\
&+\bigg(\frac{C_3}{(t-s)^{\frac12}\rho^{n(\frac1r-\frac1{mr})}}+\frac{C_4}{(t-s)^{\frac n2(\frac1r-\frac1{mr})+\frac12}}\bigg)\|v(s)\|^m_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\\
\le&\ 2\bigg(\frac{C_3}{(t-s)^{\frac12}\rho^{n(\frac1r-\frac1{mr})}}+\frac{C_4}{(t-s)^{\frac n2(\frac1r-\frac1{mr})+\frac12}}\bigg)\|v(s)\|^m_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}
\end{aligned}
$$

Consequently, by the Lebesgue convergence hypothesis, we have $I_1\to 0$ as $h\to 0$.

$$
\begin{aligned}
I_2\le&\int_t^{t+h}\bigg(\frac{C_3}{(t+h-s)^{\frac12}\rho^{\frac n2(\frac1r-\frac1{mr})}}+\frac{C_4}{(t+h-s)^{\frac n2(\frac1r-\frac1{mr})+\frac12}}\bigg)\|v(s)\|^m_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}ds\\
\le&C\big(h^{\frac12}\rho^{-\frac n2(\frac1r-\frac1{mr})}+h^{\frac12-\frac n2(\frac1r-\frac1{mr})}\sup_{0<t<T}\|v(t)\|^m_{L^{mr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\to 0 \text{ as } h\to 0.
\end{aligned}
$$

This indicates that $v\in C((0,T);\mathcal{L}^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n))$. This finishes the evidence of Theorem 1.1.
□

As t approaches to zero, we take into account the convergence of mild solutions to initial condition. The portrayal of $\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ in Proposition 2.3 is basically applied. We will demonstrate that as t approaches to zero, $v(t)$ converges to v_0 in $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-norm if v_0 belongs to $\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.

Proof of Theorem 1.2. From (4.4) it is simple to confirm

$$\begin{aligned} &\left\| \int_0^t b\cdot\nabla e^{(t-s)\Delta}\big(|v(s)|^{m-1}v(s)\big)\,ds\right\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\leq Ct^{\frac{1}{2}-\frac{m\alpha}{2}} \to 0 \text{ as } t\to 0. \end{aligned} \tag{4.17}$$

For every initial data $v_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, we have from (4.17) and Propositin 2.3,

$$\lim_{t\to 0}\|v(t)-v_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \leq \lim_{t\to 0}\|e^{t\Delta}v_0 - v_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} = 0.$$

❑

5. Conclusion

In this paper, we consider existence and uniqueness issue for a convection–diffusion equation in uniformly local Lebesgue spaces. In uniformly local Lebesgue spaces, we established the local existence and uniqueness of the mild solution for a convection–diffusion equation. These results provide new insights into the behaviour of solution of the convection-diffusion equation in uniformly local Lebesgue spaces and have implications for the modelling and simulation of complex physical systems.

References

[1] J. Aguirre, M. Escobedo, E. Zuazua, *Self-similar solutions of a convection diffusion equation and related elliptic problems*, Comm. Partial Differential Equations, **15** (1990), 139–157.

[2] J. M. Arrieta, J. W. Cholewa, T. Dlotko, A. Rodríguez-Bernal, *Dissipative parabolic equations in locally uniform spaces* , Math. Nachr., **280** (2007), 1643 –1663.

[3] J. M. Arrieta, J. W. Cholewa, T. Dlotko, A. Rodríguez-Bernal, *Linear parabolic equations in locally uniform spaces*, Math. Models Methods Appl. Sci., **14** (2004), 253–293.

[4] H. Brezis, T. Cazenave, *A nonlinear heat equation with singular initial data*, J. Anal. Math. **68** (1996), 277–304.

[5] J. W. Cholewa, A. Rodríguez-Bernal, *Extremal equilibria for dissipative parabolic equations in locally uniform spaces*, Math. Models Methods Appl. Sci., **19** (2009), 1995–2037.

[6] H. J. Choi , W. Choi, Y. Koh, *Convergence analysis of the splitting method to the nonlinear heat equation*, Num. Meth. Partial Differential Equations, **39** (2023),3417 –3440.

[7] G. Duro, E. Zuazua, *Large time behavior for convection-diffusion equations in $\mathbb{R}^N$ with periodic coefficients* , J. Differential Equations, **167** (2000), 275–315.

[8] M. Escobedo, E. Zuazua, *Large time behavior for convection-diffusion equations in* $\mathbb{R}^N$, J. Funct. Anal., **100** (1991), 119–161.

[9] M. Escobedo, J. L. Vázquez, E. Zuazua, *Entropy solutions for diffusion-convection equations with partial diffusivity*, Trans. Amer. Math. Soc., **343** (1994), 829–842.

[10] M. Escobedo, J. L. Vázquez, E. Zuazua, *Asymptotic behaviour and source-type solutions for a diffusion-convection equation*, Arch. Rational Mech. Anal., **124** (1993), 43–65.

[11] M. Escobedo, J. L. Vázquez, E. Zuazua, *A diffusion-convection equation in several space dimensions*, Indiana Univ. Math. J., **42** (1993), 1413–1440.

[12] Y. Fujishima, N. Ioku, *Existence and nonexistence of solutions for the heat equation with a superlinear source term*, J. Math. Pures Appl., 118 (2018), 128–158.

[13] K. Hisa, K. Ishige, J. Takahashi, *Existence of solutions for an inhomogeneous fractional semilinear heat equation*, Nonlinear Anal. **199** (2020), 111920.

[14] R. E. Grundy, *Asymptotic solutions of a model nonlinear convective diffusion equation*, IMA J. Appl. Math. **31** (1983), 121–137.

[15] Y. Giga, K. Inui, S. Matsui, *On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data*, Advances in fluid dynamics 4, Quad. Mat., Dept. Math., Seconda Univ. Napoli, Caserta,(1999) 27–68.

[16] M. R. Haque, T. Ogawa, R. Sato, *Existence of weak solutions to a convection–diffusion equation in a uniformly local Lebesgue space*, Commun. Pure Appl. Anal., **19** (2020), 677–697.

[17] N. Ioku, T. Ogawa, *Critical dissipative estimate for a heat semigroup with a quadratic singular potential and critical exponent for nonlinear heat equations*, J. Differential Equations, **266** (2019), 2274–2293.

[18] N. Ioku, B. Ruf, E. Terraneo, *Existence, Non-existence, and Uniqueness for a Heat Equation with Exponential Nonlinearity in* $\mathbb{R}^2$, Mathematical Physics Anal. and Geom., **18** (2015), 120–138.

[19] K. Ishige, R. Sato, *Heat equation with a nonlinear boundary condition and uniformly local* L^r *spaces*, Discrete Contin. Dyn. Syst., **36** (2016), 2627–2652.

[20] T. Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Arch. Rat. Mech. Anal., **58** (1975), 181–205.

[21] G. F. Lu, H. M. Yin, *Sourc-type solutions of heat euations with convection in several variables space*, Sci China Math, **56** (2011), 1145–1173.

[22] Y. Maekawa, Y. Terasawa, *The Navier-Stokes equations with initial data in uniformly local* L^p *spaces*, Differential Integral Equations, **19** (2006), 369–400.

[23] M. Majdour, S. Tayachi, *Global Existence and Decay Estimates for the Heat Equation with Exponential Nonlinearty*, Funkcialaj Ekvacioj, **64** (2021), 237–259.

[24] D. Mukherjee, P. T. Nam, P. Nguyen *Uniqueness of ground state and minimal-mass blow-up solutions for focusing NLS with Hardy potential*, J. Funct. Anal., **281** (2021), 109–092.

[25] P. Quittner and P. Souplet, *Superlinear Parabolic Problems, Blow-up, Global Existence and Steady States*, Birkhauser Advanced Texts: Basler Lehrbucher Birkhauser Verlag, Basel, 2007.

[26] F. B. Weissler, *Existence and non-existence of global solutions for a semilinear heat equation*, Israel J. Math., **38** (1981), 29–40.

[27] E. Zuazua, *A dynamical system approach to the self similar large time behavior in scalar convection-diffusion equations*, J. Differential Equations, **108** (1994), 1–35.