CONVECTION–DIFFUSION EQUATION IN UNIFORMLY LOCAL LEBESGUE SPACES

ABSTRACT. In this paper we establish the local existence and uniqueness of the mild solution of the Cauchy problem for convection–diffusion equation in the n-dimensional Euclidean space with initial data in uniformly local function spaces $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. For the proof, we use the $L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$- $L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimate for the convolution operators obtained by Maekawa and Terasawa [18], and the Banach fixed point theorem.

1. Introduction

We consider the Cauchy problem for the convection–diffusion equation in $\mathbb{R}^n$. For $a \in \mathbb{R}^n \setminus \{0\}$ and $p > 1$,

$$\begin{cases} \partial_t u - \Delta u = a \cdot \nabla(|u|^{p-1}u), & t > 0,\, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u = u(t,x)$; $\mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is an unknown function, and $u_0 = u_0(x)$; $\mathbb{R}^n \to \mathbb{R}$ is the initial condition.

Our main purpose here is to solve (1.1) and show the well-posedness for initial data which may not decay at space infinity but not necessarily be locally bounded.

Escobedo and Zuazua [7] showed that, for any initial data $u_0 \in L^1(\mathbb{R}^n)$, there exists a unique global classical solution $u \in C([0,\infty); L^1(\mathbb{R}^n))$ of (1.1) with

$$u \in C((0,\infty); W^{2,q}(\mathbb{R}^n)) \cap C^1((0,\infty); L^q(\mathbb{R}^n)), \tag{1.2}$$

for any $q \in (1,\infty)$. They also described the large–time behavior of solutions to (1.1) and showed decay properties when the initial data is in $L^1(\mathbb{R}^n)$.

Problem (1.1) has been considered by many authors (see, e.g., [1], [5], [6], [7], [8], [9], [10], [11], [12], [17], [21], [22]).

On the other hand, in [2], [3], [4], [15], [16], [18] and [19], the authors make use of spaces of functions which have the property that their elements have some uniform size when it is measured in balls of fixed radius but arbitrary center. These spaces are called as uniformly local spaces. These spaces are natural and useful for finding the solutions of parabolic equations in unbounded domains with non-decaying initial functions. The

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spaces enjoy suitable inclusion properties and have locally compact embeddings and besides any constant functions belong to them. In particular, the definition of uniformly local Lebesgue space is very simple and it obviously contains some functions which may have singularities and may not decay at space infinity. Moreover, the convergences of mild solutions to initial data when time goes to zero are relatively simple. Maekawa and Terasawa [18] constructed mild solution of Navier-Stokes equations with initial data in uniformly local Lebesgue spaces and they established the $L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimates for convolution kernels including $e^{t\Delta}$, $\nabla e^{t\Delta}$ and $e^{t\Delta}\mathbf{P}\nabla$. The existence and uniqueness of weak solutions of (1.1) was shown Haque–Ogawa–Sato [14] with initial data in uniformly local function spaces $\mathcal{L}^r_{\mathrm{uloc},\rho}(\Omega)$. To this end, they introduced the solution obtained by the semigroup method in $BUC(\Omega)$, bounded uniformly continuous functions. In this paper, we extend the result included in [14] into uniformly local Lebesgue space.

Definition (Uniformly local Lebesgue spaces). Let $1 \le r \le \infty$ and $\rho > 0$. The uniformly local Lebesgue spaces on $\mathbb{R}^n$ denoted by $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, is defined by

$$L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \ \|f\|_{L^r_{\mathrm{uloc},\rho}} < \infty \right\},$$

where for $\rho > 0$

$$\|f\|_{L^r_{\mathrm{uloc},\rho}} = \begin{cases} \sup\limits_{x\in\mathbb{R}^n} \left(\displaystyle\int_{B_\rho(x)} |f(y)|^r dy \right)^{\frac{1}{r}}, & 1 \le r < \infty, \\ \sup\limits_{x\in\mathbb{R}^n} \sup\limits_{y\in B_\rho(x)} |f(y)|, & r = \infty. \end{cases} \tag{1.3}$$

Here we identify $L^\infty_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ as $L^\infty(\mathbb{R}^n)$. The space $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ is a Banach space with the norm defined in (1.3). We define the subspace $\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ as the closure of the space of bounded uniformly continuous functions $BUC(\mathbb{R}^n)$ in the space $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, i.e.,

$$\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n) := \overline{BUC(\mathbb{R}^n)}^{\|\cdot\|_{L^r_{\mathrm{uloc},\rho}}}$$

and define $\mathcal{L}^\infty_{\mathrm{uloc},\rho}(\mathbb{R}^n) = BUC(\mathbb{R}^n)$.

To solve (1.1) we convert the equations to the integral equation of the form

$$u(t) = e^{t\Delta}u_0 + \int_0^t a \cdot \nabla e^{(t-s)\Delta}\big(|u(s)|^{p-1}u(s)\big)ds. \tag{1.4}$$

Here $e^{t\Delta}u_0$ is the heat semigroup. The solution of the integral equation (1.4) is called the mild solution of (1.1) with initial data u_0. The precise meaning of each term and the fact that they are well defined follows from the $L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ type estimates of convolution type operators obtained in [3] and [18].

We state our main result for the existence of a mild solution to (1.1) in uniformly local Lebesgue spaces.

Theorem 1.1 (Existence and uniqueness)**.** *Let $p > 1$ and $1 \le r < \infty$ with*

$$\begin{cases} r > n(p-1) & \text{if} \quad p > 1 + \frac{1}{n}, \\ r > 1 & \text{if} \quad p = 1 + \frac{1}{n}, \\ r \ge 1 & \text{if} \quad 1 < p < 1 + \frac{1}{n}. \end{cases} \tag{1.5}$$

Then, for any $u_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, there exist a positive T, depending only on n, p, r and $\|u_0\|_{L^r_{\mathrm{uloc},\rho}}$ and a unique mild solution $u \in L^\infty(0,T;\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)) \cap C((0,T);\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$ of (1.1).

Theorem 1.2 (Convergence to initial data)**.** *Assume the same conditions as in Theorem 1.1. Let $u \in L^\infty(0,T;\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$ be a unique mild solution with initial data $u_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. Then we have*

$$\lim_{t \to 0} \|u(t) - u_0\|_{L^r_{\mathrm{uloc},\rho}} = 0. \tag{1.6}$$

This paper is organized as follows. In section 2, we will state some properties of uniformly local spaces. In section 3, we will state $L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ type estimates of convolution operators with integrable functions satisfying some conditions. By using these estimates, we will construct mild solutions of (1.1). In section 4, we will prove our main Theorem 1.1 and Theorem 1.2 using the estimates that stated in section 3.

2. Uniformly local spaces and their properties

In this section, we state several properties of the function spaces $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ and $\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. When $\rho = 1$, we abbreviate the notation such as $L^r_{\mathrm{uloc}}(\mathbb{R}^n) := L^r_{\mathrm{uloc},1}(\mathbb{R}^n)$ and $\mathcal{L}^r_{\mathrm{uloc}}(\mathbb{R}^n) := \mathcal{L}^r_{\mathrm{uloc},1}(\mathbb{R}^n)$. The inclusion relations for $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ spaces are as follows.

Proposition 2.1. (1) *For any $\rho_1, \rho_2 > 0$, we have $L^r_{\mathrm{uloc},\rho_1}(\mathbb{R}^n) = L^r_{\mathrm{uloc},\rho_2}(\mathbb{R}^n)$ with equivalents norm.*
(2) *For any $1 \le p \le q \le \infty$ and $\rho > 0$, we have $L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n) \subset L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.*
(3) *Let $1 \le r < \infty$ and $\rho > 0$, we have $L^r(\mathbb{R}^n) \subset L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, $L^\infty(\mathbb{R}^n) \subset L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.*

Proof of Proposition 2.1. The inclusions (1), (2) and (3) of the Proposition 2.1 follow from the Hölder inequality, we omit the detail. ❑

Proposition 2.2. *The class of compact supported smooth functions; $C_0^\infty(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ are not dense in $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.*

Proof of Proposition 2.2. We know that $1 \in L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. Let $f \in C_0^\infty(\mathbb{R}^n)$. We may assume that there exist $R_f > 0$ such that $\operatorname{supp} f \subset B_{R_f}(0)$. Then

$$\|f - 1\|_{L^r_{\mathrm{uloc},\rho}} = \sup_{x \in \mathbb{R}^n} \left(\int_{B_\rho(x)} |f(y) - 1|^r dy \right)^{\frac{1}{r}}$$
$$\ge \left(\int_{B_\rho(x))} |f(y) - 1|^r dy \right)^{\frac{1}{r}}.$$

Take $x_0 \in \mathbb{R}^n$ such that $|x_0| > R_f + \rho$. Then $f(y) = 0$ on $B_\rho(x_0)$.
Now

$$\begin{aligned}\|f-1\|_{L^r_{\mathrm{uloc},\rho}} \geq & \left(\int_{B_\rho(x_0)} 1dy\right)^{\frac{1}{r}} \\ =& |B_\rho(x_0)|^{\frac{1}{r}} = |B_\rho(0)|^{\frac{1}{r}}.\end{aligned}$$

This holds for any $f \in C_0^\infty(\mathbb{R}^n)$. Hence for all $f \in C_0^\infty(\mathbb{R}^n)$ can not approximates $1 \in L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. This stands that $C_0^\infty(\mathbb{R}^n)$ is not dense in $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.

Let $f \in C_0^\infty(\mathbb{R}^n)$ be a function such that *supp* $f \subset B_1(0)$, $\int_{\mathbb{R}^n} |f|^r dx = 1$ and $f \geq 0$. Choose any countable points $\{x_m\}_m \geq 1$ such that $B_1(x_m) \cap B_1(x_{m'}) = \emptyset$. Set $f_m(x) = m^{\frac{n}{r}} f(m(x - x_m))$. Then *supp* $f_m \subset B_{\frac{1}{m}}(x_m)$. So if we set

$$\tilde{f}(x) = \begin{cases} f_m(x), & \text{for } \ x \in B_1(x_m), \\ 0, & \text{otherwise,} \end{cases}$$

then we have

$$\|\tilde{f}\|_{L^r_{\mathrm{uloc}}(\mathbb{R}^n)} = \sup_m \|f_m\|_{L^r(B_1(x_m))} = 1.$$

Let $q > r$. For any $g \in L^r_{\mathrm{uloc}}(\mathbb{R}^n)$, we have $\|\tilde{f} - g\|L^r_{\mathrm{uloc}}(\mathbb{R}^n) \geq 1$. Indeed,

$$\begin{aligned}\|\tilde{f}-g\|_{L^r_{\mathrm{uloc}}(\mathbb{R}^n)} &\geq \|f_m - g\|_{L^r(B_1(x_m))} \geq \|f_m - g\|_{L^r(B_{\frac{1}{m}}(x_m))} \\ &\geq \left| \|f_m\|_{L^r(B_{\frac{1}{m}}(x_m))} - \|g\|_{L^r(B_{\frac{1}{m}}(x_m))} \right| \geq 1 - |B_{\frac{1}{m}}(x_m)|^{\frac{1}{r}(1-\frac{r}{q})} \|g\|_{L^q_{\mathrm{uloc}}(\mathbb{R}^n)} \\ &\geq 1 - \frac{C}{m^{n(\frac{1}{r}-\frac{1}{q})}} \|g\|_{L^q_{\mathrm{uloc}}(\mathbb{R}^n)} \to 1 \quad \text{as} \quad m \to \infty.\end{aligned}$$

This implies that $\tilde{f}$ does not belong to $\overline{\cup_{q>r} L^q_{\mathrm{uloc}}(\mathbb{R}^n)}^{L^r_{\mathrm{uloc}}(\mathbb{R}^n)}$. Therefore, we have that the subset $\cup_{q>r} L^q_{\mathrm{uloc}}(\mathbb{R}^n)$ is not dense in $L^r_{\mathrm{uloc}}(\mathbb{R}^n)$ and implies that $L^\infty(\mathbb{R}^n)$ is not dense in $L^r_{\mathrm{uloc}}(\mathbb{R}^n)$. Hence the space $L^\infty(\mathbb{R}^n)$ not dense in $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. ❑

We have the following characterizations of $\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. These characterizations are obtained in [3] and [18].

Proposition 2.3. *For any $\rho > 0$, the following three properties are equivalent:*
(1) $f \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.
(2) $\lim_{|y|\to 0} \|f(\cdot + y) - f(\cdot)\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} = 0$.
(3) $\lim_{t\to 0+} \|e^{t\Delta} f - f\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} = 0$.

For the proof see [18].

3. Mild solutions in uniformly local spaces

In this section, we will give mild solutions, i.e., solutions of the integral equations for the convection–diffusions equation (1.1). And we will also state the key estimates for the convolution operators $e^{t\Delta}$ and $\nabla e^{t\Delta}$ which are obtained in [3] and [18].

Definition (Mild solutions). Let $p > 1$ and $T > 0$. The function $u \in L^\infty(0,T;\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$ is called a mild solution of the convection–diffusion equations (1.1) on $(0,T)\times\mathbb{R}^n$ if there exist $u_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ such that

$$u(t) = e^{t\Delta}u_0 + \int_0^t a\cdot\nabla e^{(t-s)\Delta}\big(|u(s)|^{p-1}u(s)\big)ds \tag{3.1}$$

holds in $C((0,T);\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$.

The following theorem is obtained in [18].

Theorem 3.1. *Let $1 \le q \le p \le \infty$ and G_t is the heat kernel in $\mathbb{R}^n$. We set $G_t(x) = t^{-\frac{n}{2}}G_1(\frac{x}{\sqrt{t}})$ for $t > 0$. Then, for any function $f \in L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, we can define pointwise*

$$G_t * f(x) = \int_{\mathbb{R}^n} G_t(x-y)f(y)dy.$$

Furthermore, we have the estimate

$$\|G_t * f\|_{L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \le \left(\frac{C_1\|G_1\|_{L^1(\mathbb{R}^n)}}{\rho^{n(\frac{1}{q}-\frac{1}{p})}} + \frac{C_2\|G_1\|_{L^r(\mathbb{R}^n)}}{t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}}\right)\|f\|_{L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)}, \tag{3.2}$$

where r is the number satisfying $\frac{1}{p} = \frac{1}{r}+\frac{1}{q}-1$, and C_1, C_2 are positive constants depending only on n.

For the proof see [18].

From Theorem 3.1, we can establish $L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimates for the linear and nonlinear term in the integral equation (3.1).

Corollary 3.2 ($L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimates)**.** *Let $1 \le q \le p \le \infty$. Then for any $f \in L^p_{uloc,\rho}(\mathbb{R}^n)$, we have*

$$\|e^{t\Delta}f\|_{L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \le \left(\frac{C_1}{\rho^{n(\frac{1}{q}-\frac{1}{p})}} + \frac{C_2}{t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}}\right)\|f\|_{L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)}, \tag{3.3}$$

$$\|\nabla e^{t\Delta}f\|_{L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \le \left(\frac{C_3}{t^{\frac{1}{2}}\rho^{n(\frac{1}{q}-\frac{1}{p})}} + \frac{C_4}{t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}}}\right)\|f\|_{L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)}, \tag{3.4}$$

holds. Here C_1, C_3 are positive constants depending only on n and C_2, C_4 are positive constants depending only on n, p and q.

For the proof see [18].

Remark: The $L^p_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimates for heat semigroup are also obtained in [3]. When $p = q = \infty$, the estimate is obtained in [13].

4. Proof of Theorem

Proof of Theorem 1.1. For $M > 0$, $T > 0$ and $1 \le r < \infty$, we let

$$X = L^\infty(0,T;L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)) \cap L^\infty_{loc}(0,T;L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n))$$

and

$$X_{M,T,r} := \Big\{ u \in X : \sup_{0<t<T} \|u(t)\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \le M \quad \text{and} \quad \sup_{0<t<T} t^{\frac{\alpha}{2}} \|u(t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \le M \Big\},$$

where $\alpha = \frac{n(p-1)}{pr} < \frac{1}{p} < 1$ and M and T are constants depending on $\|u_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)}$, r, p and n, determined later. $X_{M,T,r}$ is a complete metric space with the metric; for any u and $v \in X_{M,T,r}$,

$$d_{X_{M,T,r}}(u,v) = \|u - v\|_{X_{M,T,r}} \equiv \sup_{0<t<T} t^{\frac{\alpha}{2}} \|u(t) - v(t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}.$$

Then we consider a map

$$\Phi : X_{M,T,r} \to X_{M,T,r} \qquad \text{by}$$

$$\Phi[u](t) = e^{t\Delta} u_0 + \int_0^t a \cdot \nabla e^{(t-s)\Delta} \big(|u(s)|^{p-1} u(s) \big) ds, \qquad u_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n), \tag{4.1}$$

and show that Φ is a contraction mapping from $X_{M,T,r}$ onto itself. This implies an existence of the fixed point for the map Φ on $X_{M,T,r}$ and it becomes a solution to the coreesponding integral equation (3.1) has a unique fixed point and it becomes an $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-mild solution.

For any $u \in X_{M,T,r}$, we will estimate $\|\Phi[u](t)\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)}$ and $t^{\frac{\alpha}{2}} \|\Phi[u](t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}$. By taking $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimate, we obtain

$$\begin{aligned} \|e^{t\Delta} u_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} &\le \left(\frac{C_1}{\rho^{n(\frac{1}{r} - \frac{1}{r})}} + \frac{C_2}{t^{\frac{n}{2}(\frac{1}{r} - \frac{1}{r})}} \right) \|u_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &= C \|u_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)}, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\big\| \nabla e^{(t-s)\Delta} \big(|u(s)|^{p-1} u(s) \big) \big\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\qquad \le \left(\frac{C_3}{(t-s)^{\frac{1}{2}} \rho^{n(\frac{1}{r} - \frac{1}{r})}} + \frac{C_4}{(t-s)^{\frac{n}{2}(\frac{1}{r} - \frac{1}{r}) + \frac{1}{2}}} \right) \| |u(s)|^{p-1} u(s) \|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\qquad = C'(t-s)^{-\frac{1}{2}} \|u(s)\|^p_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}. \end{aligned} \tag{4.3}$$

Then, we obtain

$$\begin{aligned} &|a| \int_0^t \big\| \nabla e^{(t-s)\Delta} \big(|u(s)|^{p-1} u(s) \big) \big\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} ds \\ &\qquad \le C' \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|^p_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} ds \\ &\qquad \le C' \big(\sup_{0<t<T} t^{\frac{\alpha}{2}} \|u(t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \big)^p \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{p\alpha}{2}} ds \\ &\qquad \le C' M^p t^{\frac{1}{2} - \frac{p\alpha}{2}} \beta\big(\frac{1}{2}, 1 - \frac{p\alpha}{2} \big). \end{aligned} \tag{4.4}$$

The integral and $t^{\frac{1}{2}-\frac{p\alpha}{2}}$ is bounded if and only if $1-\frac{p\alpha}{2}>0$ and $\frac{1}{2}-\frac{p\alpha}{2}\geq 0$, that is, $r>n(p-1)$. Thus if these conditions are satisfied, we have

$$\|\Phi[u](t)\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \leq C\|u_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} + C'M^pT^{\frac{1}{2}-\frac{p\alpha}{2}}\beta(\frac{1}{2}, 1-\frac{p\alpha}{2}). \tag{4.5}$$

Next we estimate $t^{\frac{\alpha}{2}}\|\Phi[u](t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}$. By taking $L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimate and choose $\rho\geq t^{\frac{1}{2}}$, we obtain

$$\begin{aligned} t^{\frac{\alpha}{2}}\|e^{t\Delta}u_0\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} &\leq t^{\frac{\alpha}{2}}\left(\frac{C_1}{\rho^{n(\frac{1}{r}-\frac{1}{pr})}} + \frac{C_2}{t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})}}\right)\|u_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\leq t^{\frac{\alpha}{2}}t^{-\frac{\alpha}{2}}\|u_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} = C\|u_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)}, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\left\|\nabla e^{(t-s)\Delta}\left(|u(s)|^{p-1}u(s)\right)\right\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\leq \left(\frac{C_3}{(t-s)^{\frac{1}{2}}\rho^{n(\frac{1}{r}-\frac{1}{pr})}} + \frac{C_4}{(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})+\frac{1}{2}}}\right)\||u(s)|^{p-1}u(s)\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &= C'(t-s)^{-\frac{1}{2}-\frac{\alpha}{2}}\|u(s)\|^p_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}. \end{aligned} \tag{4.7}$$

Then, we obtain

$$\begin{aligned} &|a|\ t^{\frac{\alpha}{2}}\int_0^t \left\|\nabla e^{(t-s)\Delta}\left(|u(s)|^{p-1}u(s)\right)\right\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}ds \\ &\qquad\leq C't^{\frac{\alpha}{2}}\int_0^t (t-s)^{-\frac{1}{2}-\frac{\alpha}{2}}\|u(s)\|^p_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}ds \\ &\qquad\leq C't^{\frac{\alpha}{2}}\big(\sup_{0<t<T} t^{\frac{\alpha}{2}}\|u(t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\big)^p\int_0^t (t-s)^{-\frac{1}{2}-\frac{\alpha}{2}}s^{-\frac{p\alpha}{2}}ds \\ &\qquad\leq C'M^pt^{\frac{1}{2}-\frac{p\alpha}{2}}\beta\big(\frac{1}{2}-\frac{\alpha}{2}, 1-\frac{p\alpha}{2}\big). \end{aligned} \tag{4.8}$$

The integral and $t^{\frac{1}{2}-\frac{p\alpha}{2}}$ is bounded if and only if $\frac{1}{2}-\frac{\alpha}{2}>0$, $1-\frac{p\alpha}{2}>0$ and $\frac{1}{2}-\frac{p\alpha}{2}\geq 0$, that is, $\alpha<1$ and $r>n(p-1)$. Thus if these conditions are satisfied, we have

$$t^{\frac{\alpha}{2}}\|\Phi[u](t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \leq C\|u_0\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} + C'M^pT^{\frac{1}{2}-\frac{p\alpha}{2}}\beta(\frac{1}{2}-\frac{\alpha}{2}, 1-\frac{p\alpha}{2}). \tag{4.9}$$

We next consider the condition in which Φ is contraction. Let u_0, $v_0\in\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. For any u, $v\in X_{M,T,r}$ and $t\in(0,T)$, we have

$$\begin{aligned} t^{\frac{\alpha}{2}}\|\Phi[u](t)-\Phi[v](t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} &\leq t^{\frac{\alpha}{2}}\|e^{t\Delta}(u_0-v_0)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &+|a|t^{\frac{\alpha}{2}}\int_0^t \left\|\nabla e^{(t-s)\Delta}\left(|u(s)|^{p-1}u(s)-|v(s)|^{p-1}v(s)\right)\right\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}ds. \end{aligned} \tag{4.10}$$

By taking $L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimate and choose $\rho \geq t^{\frac{1}{2}}$, we obtain

$$\begin{aligned} t^{\frac{\alpha}{2}}\|e^{t\Delta}u_0 - e^{t\Delta}v_0\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} &\leq t^{\frac{\alpha}{2}}\left(\frac{C_1}{\rho^{n(\frac{1}{r}-\frac{1}{pr})}} + \frac{C_2}{t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})}}\right)\|u_0 - v_0\|_{L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\leq Ct^{\frac{\alpha}{2}}t^{-\frac{\alpha}{2}}\|u_0 - v_0\|_{L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &= C\|u_0 - v_0\|_{L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}. \end{aligned} \tag{4.11}$$

Again, by taking $L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimate, choose $\rho \geq (t-s)^{\frac{1}{2}}$ and then applying the Hölder inequality, we obtain

$$\begin{aligned} &\left\|\nabla e^{(t-s)\Delta}\big(|u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s)\big)\right\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\quad\leq \left(\frac{C_3}{(t-s)^{\frac{1}{2}}\rho^{n(\frac{1}{r}-\frac{1}{pr})}} + \frac{C_4}{(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})+\frac{1}{2}}}\right)\left\||u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s)\right\|_{L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\quad\leq \left(\frac{C_3}{(t-s)^{\frac{1}{2}}\rho^{n(\frac{1}{r}-\frac{1}{pr})}} + \frac{C_4}{(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})+\frac{1}{2}}}\right) \\ &\qquad\qquad \left\|\big(p\max\{|u(s)|^{p-1}, |v(s)|^{p-1}\}|u(s)-v(s)|\big)\right\|_{L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\quad\leq \left(\frac{C_3}{(t-s)^{\frac{1}{2}}(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})}} + \frac{C_4}{(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})+\frac{1}{2}}}\right) \\ &\qquad\qquad \left\|\big(p\max\{|u(s)|^{p-1}, |v(s)|^{p-1}\}|u(s)-v(s)|\big)\right\|_{L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\quad\leq C'(t-s)^{-\frac{1}{2}-\frac{\alpha}{2}}\|u(s)-v(s)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\left\|\max\{|u(s)|^{p-1}, |v(s)|^{p-1}\}\right\|_{L^{\frac{pr}{p-1}}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\quad\leq C'(t-s)^{-\frac{1}{2}-\frac{\alpha}{2}}\|u(s)-v(s)\|_{L^{pr}_{\mathrm{uloc},\rho}}\big(\|\,|u(s)|^{p-1}\|_{L^{\frac{pr}{p-1}}_{\mathrm{uloc},\rho}} + \|\,|v(s)|^{p-1}\|_{L^{\frac{pr}{p-1}}_{\mathrm{uloc},\rho}}\big) \\ &\quad= C'(t-s)^{-\frac{1}{2}-\frac{\alpha}{2}}\|u(s)-v(s)\|_{L^{pr}_{\mathrm{uloc},\rho}}\big(\|u(s)\|^{p-1}_{L^{pr}_{\mathrm{uloc},\rho}} + \|v(s)\|^{p-1}_{L^{pr}_{\mathrm{uloc},\rho}}\big). \end{aligned} \tag{4.12}$$

Hence we have

$$\begin{aligned} &\sup_{t\in(0,T)} t^{\frac{\alpha}{2}}\|\Phi[u](t) - \Phi[v](t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\ &\quad\leq C\|u_0 - v_0\|_{L^{r}_{uloc,\rho}} + C'M^{p-1}\sup_{t\in(0,T)} t^{\frac{\alpha}{2}}\|u(t)-v(t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\int_0^t (t-s)^{-\frac{1}{2}-\frac{\alpha}{2}}s^{-\frac{\alpha}{2}(p-1)}ds \\ &\quad\leq C\|u_0 - v_0\|_{L^{r}_{uloc,\rho}} + C'M^{p-1}t^{\frac{1}{2}-\frac{\alpha p}{2}}\sup_{t\in(0,T)} t^{\frac{\alpha}{2}}\|u(t)-v(t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\beta\Big(\frac{1}{2}-\frac{\alpha}{2}, 1-\frac{\alpha(p-1)}{2}\Big). \end{aligned} \tag{4.13}$$

The integral and $t^{\frac{1}{2}-\frac{p\alpha}{2}}$ is bounded if and only if $\frac{1}{2}-\frac{\alpha}{2}>0$, $1-\frac{(p-1)\alpha}{2}>0$ and $\frac{1}{2}-\frac{p\alpha}{2}\geq 0$, that is, $\alpha<1$ and $r>n(p-1)$. Thus if these conditions are satisfied, we have

$$\begin{aligned}&\sup_{t\in(0,T)} t^{\frac{\alpha}{2}}\|\Phi[u](t)-\Phi[v](t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\\ &\qquad\leq C\|u_0-v_0\|_{L^r_{uloc,\rho}}+C'M^{p-1}t^{\frac{1}{2}-\frac{\alpha p}{2}}\sup_{t\in(0,T)} t^{\frac{\alpha}{2}}\|u(t)-v(t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\\ &\qquad\leq C\|u_0-v_0\|_{L^r_{uloc,\rho}}+C'M^{p-1}T^{\frac{1}{2}-\frac{\alpha p}{2}}\sup_{t\in(0,T)} t^{\frac{\alpha}{2}}\|u(t)-v(t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}.\end{aligned}\tag{4.14}$$

Setting $u_0=v_0$, we obtain

$$d_{X_{M,T,r}}(\Phi[u],\Phi[v])\leq CM^{p-1}T^{\frac{1}{2}-\frac{\alpha p}{2}}d_{X_{M,T,r}}(u,v).$$

It follows from the above estimates that if T is small enough then the map Φ is contraction from $X_{M,T,r}$ onto $X_{M,T,r}$ and by virtue of the Banach fixed piont theorem, there exist a unique fixed point of Φ in $X_{M,T,r}$. By the definition, this fixed point satisfies the integral equation (3.1) and besides, $u(t)\to u_0$ as $t\to 0$ by Theorem 1.2. Hence u is the $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-mild solution to (1.1). This shows the existence of solution.

Next we show the uniqueness of the mild solution in $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.
Let u_1, $u_2\in L^\infty(0,T;\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$ be two mild solutions with initial data $u_0\in\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. Then by (4.14), for any $0<t<T'<T$, we obtain

$$\begin{aligned}&\sup_{t\in(0,T')} t^{\frac{\alpha}{2}}\|u_1(t)-u_2(t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\\ &\qquad\leq CM^{p-1}T'^{\frac{1}{2}-\frac{\alpha p}{2}}\sup_{t\in(0,T')} t^{\frac{\alpha}{2}}\|u_1(t)-u_2(t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}.\end{aligned}$$

Thus, for sufficently small $T'>0$ it follows that $u_1=u_2$ in $0<t<T'$. Repeating this argument, we see that $u_1=u_2$ in $0<t<T$.

We will show $u\in C((0,T);\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$. Let $0<t<t+h<T$. Since

$$u(t)=e^{t\Delta}u_0-\int_0^t a\cdot\nabla e^{(t-s)\Delta}\big(|u(s)|^{p-1}u(s)\big)ds,$$

we have

$$\begin{aligned}&\|u(t+h)-u(t)\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\\ &\leq\|e^{(t+h)\Delta}u_0-e^{t\Delta}u_0\|_{L^{pr}_{uloc,\rho}}\\ &+\left\|\int_0^{t+h} a\cdot\nabla e^{(t+h-s)\Delta}\big(|u(s)|^{p-1}u(s)\big)ds-\int_0^t a\cdot\nabla e^{(t-s)\Delta}\big(|u(s)|^{p-1}u(s)\big)ds\right\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}.\end{aligned}\tag{4.15}$$

From Corollary 3.2 we know that $e^{t\Delta}u_0\in\mathcal{L}^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ for any $t>0$. Therefore, by Proposition 2.3

$$\|e^{(t+h)\Delta}u_0-e^{t\Delta}u_0\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}=\|e^{h\Delta}e^{t\Delta}u_0-e^{t\Delta}u_0\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}\to 0 \text{ as } h\to 0.$$

Next,

$$
\begin{aligned}
&\left\| \int_0^{t+h} a\cdot\nabla e^{(t+h-s)\Delta}\big(|u(s)|^{p-1}u(s)\big)ds - \int_0^t a\cdot\nabla e^{(t-s)\Delta}\big(|u(s)|^{p-1}u(s)\big)ds \right\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
&\le |a|\int_0^t \big\|\nabla e^{(t+h-s)\Delta}\big(|u(s)|^{p-1}u(s)\big) - \nabla e^{(t-s)\Delta}\big(|u(s)|^{p-1}u(s)\big)\big\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}ds \\
&\qquad + |a|\int_t^{t+h} \big\|\nabla e^{(t+h-s)\Delta}\big(|u(s)|^{p-1}u(s)\big)\big\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}ds = I_1 + I_2.
\end{aligned}
\tag{4.16}
$$

Again, from Corollary 3.2 we know that $\nabla e^{(t-s)\Delta}\big(|u(s)|^{p-1}u(s)\big) \in \mathcal{L}^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ for $0 \le s < t$, therefore, we have from Proposition 2.3

$$
\big\|\nabla e^{(t+h-s)\Delta}\big(|u(s)|^{p-1}u(s)\big) - \nabla e^{(t-s)\Delta}\big(|u(s)|^{p-1}u(s)\big)\big\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \to 0 \text{ as } h \to 0,
$$

for $0 \le s < t$.

On the other hand, by taking $L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-$L^{r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ estimate, we obtain

$$
\begin{aligned}
&\big\|\nabla e^{(t+h-s)\Delta}\big(|u(s)|^{p-1}u(s)\big) - \nabla e^{(t-s)\Delta}\big(|u(s)|^{p-1}u(s)\big)\big\|_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
\le& \left(\frac{C_3}{(t+h-s)^{\frac{1}{2}}\rho^{n(\frac{1}{r}-\frac{1}{pr})}} + \frac{C_4}{(t+h-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})+\frac{1}{2}}} \right)\|u(s)\|^p_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
&+ \left(\frac{C_3}{(t-s)^{\frac{1}{2}}\rho^{n(\frac{1}{r}-\frac{1}{pr})}} + \frac{C_4}{(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})+\frac{1}{2}}} \right)\|u(s)\|^p_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
&\le 2\left(\frac{C_3}{(t-s)^{\frac{1}{2}}\rho^{n(\frac{1}{r}-\frac{1}{pr})}} + \frac{C_4}{(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})+\frac{1}{2}}} \right)\|u(s)\|^p_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}
\end{aligned}
$$

Hence, by the Lebesgue convergence theorem, we have $I_1 \to 0$ as $h \to 0$.

$$
\begin{aligned}
I_2 \le& \int_t^{t+h} \left(\frac{C_3}{(t+h-s)^{\frac{1}{2}}\rho^{\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})}} + \frac{C_4}{(t+h-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})+\frac{1}{2}}} \right)\|u(s)\|^p_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)}ds \\
\le& C\big(h^{\frac{1}{2}}\rho^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})} + h^{\frac{1}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{pr})}\sup_{0<t<T}\|u(t)\|^p_{L^{pr}_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \to 0 \text{ as } h \to 0.
\end{aligned}
$$

This implies that $u \in C((0,T);\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$. This completes the proof of Theorem 1.1. ❑

We consider the convergence of mild solutions to initial data as t goes to zero. The characterization of $\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ in Proposition 2.3 is essentially used. We will show that if u_0 belongs to $\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ then $u(t)$ converges to u_0 in $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$-norm as t goes to zero.

Proof of Theorem 1.2. From (4.4) it is easy to check that

$$
\begin{aligned}
&\left\| \int_0^t a\cdot\nabla e^{(t-s)\Delta}\big(|u(s)|^{p-1}u(s)\big)ds \right\|_{L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)} \\
&\le Ct^{\frac{1}{2}-\frac{p\alpha}{2}} \to 0 \text{ as } t \to 0.
\end{aligned}
\tag{4.17}
$$

For any $u_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, we have from (4.17) and Propositin 2.3,

$$\lim_{t\to 0}\|u(t)-u_0\|_{L^r_{\text{uloc},\rho}(\mathbb{R}^n)} \le \lim_{t\to 0}\|e^{t\Delta}u_0-u_0\|_{L^r_{\text{uloc},\rho}(\mathbb{R}^n)} = 0.$$

❑

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