The Solution of Homogeneous Liouville Fractional Differential Equations by Sumudu Transform Method

Abstract

In this paper, we study the homogeneous Liouville fractional differential equations with constant coefficients. The solutions interms of Mittag-Leffler of homogeneous Liouville fractional differential equations with constant coefficients are obtained by Sumudu transform method(STM). The results obtained by STM are illustrated by examples.

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Keywords: Sumudu transform, Mittag-Leffler functions, Wright functions, Fractional differential equations.

1 Introduction

The integral transforms are widely used in applied science,mathematical physics and engineering. In order to solve fractional differential equations, the integral transforms were extensively used and there is a lot of literature available on the theory and applications of integral transforms,such as the Laplace, Fourier, Mellin and Hankel. G. K. Watugal(1993) introduced a new integral transforms named Sumudu transform and further applied to the solution of ordinary differential equation in control engineering problems [30].

Many problems in physics, engineering and biology etc. are modeled via fractional differential equations such as diffusion, signal processing, electrochemistry, viscoelasticity [24, 27]. In literature numerous methods are available to solve fractional differential equations like power series method,iterative method, adomain decomposition method, transform method, monotone method etc.[1, 7, 9, 15, 18, 21, 26, 28]. Integral transform methods such as Fourier, Laplace, Mellin, and Hankel etc.were extensively used to study fractional differential equations [5, 6, 14, 16, 25]. In 1993, Watagulla [31, 32] introduced Sumudu transform and applied it to solve ordinary differential equations in control engineering problems. The complex inversion formula for Sumudu transform was proved by Weerakoon [28, 29] in 1994 and applied it to solve partial differential equations. Asiru studied the properties of Sumudu transform [1, 2, 3] and solved integral equations of convolution type [4] and discrete dynamical system [9]. Belgacem et al.[8] also established the properties of Sumudu transform. Kilicman et al [19, 20] successfully applied Sumudu transform method to solve system of differential equations.

Kataetbeh and Belgacem [22, 24] obtained formulae for Sumudu transform of fractional derivatives such as Riemann-Liouville , Caputo and Miller-Ross using Laplace- Sumudu duality property and obtained solutions of fractional differential equations. Bulut, Demiray and Tuluce [10, 11, 12, 13, 17] have studied heat equations and wave equations by Sumudu transform method. In this paper we apply Sumudu transform method to obtain explicit solution of homogeneous fractional differential equations with constant coefficients. We also illustrate the STM by examples.

2 Preliminary Results, Notations and Terminology

In this section we give definitions and some basic results which are used in the next section.

Definition 2.1 *[24] A real function $f(t)$, $t \geq 0$ is said to be in space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $n(> \mu)$, such that $f(t) = t^n f_1(t)$, where $f_1(t) \in C[0, \infty)$, and is said to be in space C_μ^k if and only if $f^{(k)} \in C_\mu$, $k \in \mathbb{N}$.*

Definition 2.2 *[21] The Liouville fractional integrals $I_{0+}^\alpha f$ of order α on the half-axis $\mathbb{R}^+$ is defined as*

$$(I_{0+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \quad (x > 0; \quad \Re(\alpha) > 0). \tag{2.1}$$

Definition 2.3 *[21] The Liouville fractional derivative $D_{0+}^\alpha y$ of a function $y(t)$ of order α on the half-axis $\mathbb{R}^+$ is given by*

$$\begin{aligned}(D_{0+}^\alpha y)(t) &= \left(\frac{d}{dx}\right)^n ((I_{0+}^{n-\alpha} y)(x)) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{y(t)dt}{(x-t)^{\alpha-n+1}},\end{aligned} \tag{2.2}$$

with $\quad n = [\Re(\alpha)] + 1; \Re(\alpha) \geq 0; x > 0.$

Definition 2.4 *[18] One parameter Mittag-Leffler function is denoted by $E_\alpha(z)$ and is defined as*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \ \alpha \in C, \ Re(\alpha) > 0. \tag{2.3}$$

Definition 2.5 *[4] A two-parameter Mittag-Leffler function denoted by $E_{\alpha,\beta}(z)$ and is defined as,*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ \alpha>0,\ \beta>0. \tag{2.4}$$

Definition 2.6 *[31] Consider a set A defined as*

$$A=\{f(t)|\,\exists M,\tau_1,\tau_2>0,|f(t)|\le Me^{\frac{|t|}{\tau_j}}\ \text{if}\, t\in(-1)^j\times[0,\infty)\}. \tag{2.5}$$

For all real $t\ge 0$, the Sumudu transform of a function $f(t)\in A$, denoted by $S[f(t)]=F(u)$, is defined as

$$S[f(t)](u)=F(u)=\int_0^{\infty}e^{-t}f(ut)dt,\ \ u\in(-\tau_1,\tau_2). \tag{2.6}$$

The function $f(t)$ in equation (2.6)is called the inverse Sumudu transform of $F(u)$ and is $S^{-1}[F(u)]$.

Definition 2.7 *[22] The Sumudu transform of the Liouville fractional derivative (2.2) is given by*

$$S[(D^{\alpha}_{0+}y)(t)](u)=u^{-\alpha}[Sy](u)-\sum_{j=1}^{l}d_ju^{-j},\ \ (l-1<\alpha\le l;\, l\in\mathbb{N}) \tag{2.7}$$

where

$$d_j=(D^{\alpha-j}_{0+}y)(0+),\qquad (j=1,2,...,l). \tag{2.8}$$

Theorem 2.1 *[8] Let $F(u)$ and $G(u)$ be the Sumudu transforms of $f(t)$ and $g(t)$ respectively. If*

$$h(t)=(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$$

*where * denotes convolution of f and g, then the Sumudu transform of $h(t)$ is*

$$S[h(t)]=uF(u)G(u). \tag{2.9}$$

Theorem 2.2 *[7] Let $n\ge 1$ and $F(u)$ be the Sumudu transform of the function $f(t)$. The Sumudu transform of the n^{th} derivative of $f(t)$, denoted by $S[f^{(n)}(t)](u)=F_n(u)$ is given by*

$$\begin{aligned}S[f^{(n)}(t)](u)=F_n(u)&=\frac{F(u)}{u^n}-\frac{f(0)}{u^n}-\frac{f'(0)}{u^{(n-1)}}-\ldots-\frac{f^{(n-1)}(0)}{u}\\&=\frac{F(u)}{u^n}-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{(n-k)}}.\end{aligned} \tag{2.10}$$

Lemma 2.1 *[14, 22] let $\alpha, \beta, \lambda \in \mathbb{R}$ and $\alpha > 0, \beta > 0, n \in \mathbb{N}$. Then*

$$S\left[t^{\alpha n+\beta-1}\left(\frac{\partial}{\partial\lambda}\right)^n E_{\alpha,\beta}(\lambda t^\alpha)\right] = \frac{n!u^{\alpha n+\beta-1}}{(1-\lambda u^\alpha)^{n+1}}. \tag{2.11}$$

In particular $n = 0$ and $Re(\frac{1}{u}) > \lambda$, then

$$S[t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)](u) = \frac{u^{\beta-1}}{1-\lambda u^\alpha}. \tag{2.12}$$

3 Homogenous Equations with Constant Coefficients :

In this section we apply Sumudu transform to obtain explicit solutions to the homogeneous Liouville type fractional differential equation of the form

$$\sum_{k=0}^{m} A_k(D_{0+}^{\alpha_k}y)(t) + A_0y(t) = 0 \ \ (t > 0; m \in \mathbb{N}; 0 < \alpha_1 < ... < \alpha_m) \tag{3.1}$$

The conditions when solutions $y_1(t), y_2(t), ..., y_l(t), l \in \mathbb{N}$, of equation (3.1) with $l-1 < \alpha_m < l$ will be linearly independent and these solutions form the fundamental system of solutions given by

$$(D_{o+}^{\alpha-k}y_j)(0+) = \delta_{k,j} \qquad (k, j = 1, 2, ..., l), \tag{3.2}$$

where $\delta_{k,j}$ is the Kronecker delta function.

Theorem 3.1 *Let $(t > 0; l-1 < \alpha \leq l; l \in \mathbb{N}), \lambda \in \mathbb{R}$. Then the functions*

$$y_j(t) = t^{\alpha-j}E_{\alpha,\alpha-j+1}(\lambda t^\alpha) \qquad (j = 1, 2, ..., l) \tag{3.3}$$

yield the fundamental system of solution to the equation

$$(D_{0+}^{\alpha}y)(t) - \lambda y(t) = 0. \tag{3.4}$$

Proof : Applying Sumudu transform on both sides of (3.4), we get

$$S[(D_{0+}^{\alpha}y)(t)](u) - \lambda S[y(t)](u) = 0. \tag{3.5}$$

Using (2.7)

$$u^{-\alpha}[Sy(t)](u) - \sum_{j=1}^{l} d_j u^{-j} - \lambda S[y(t)](u) = 0$$

$$S[y(t)](u) = \sum_{j}^{l} d_j \frac{u^{\alpha-j}}{(1-\lambda u^\alpha)} \tag{3.6}$$

Replacing $\beta = \alpha - j + 1$, in (2.12) and taking inverse Sumudu transform on both sides of (3.6), we get

$$y(t) = \sum_{j}^{l} d_j t^{\alpha-j} E_{\alpha,\alpha-j+1}(\lambda t^{\alpha})$$

which gives the solution of the equation (3.4) as

$$y(t) = \sum_{j=1}^{l} d_j y_j(t),$$

where

$$y_j(t) = t^{\alpha-j} E_{\alpha,\alpha-j+1}(\lambda t^{\alpha}), j = 1, 2, ..., l.$$

Example 3.1 *The equation*

$$(D_{0+}^{l-\frac{1}{2}} y)(t) - \lambda y(t) = 0, \quad (t > 0; l \in \mathbb{N}; \lambda \in \mathbb{R}) \tag{3.7}$$

has its fundamental system of solution given by

$$y_j(t) = t^{l-j-\frac{1}{2}} E_{l-\frac{1}{2},l-j+\frac{1}{2}}(\lambda t^{l-\frac{1}{2}}), \qquad (j = 1, 2, ..., l). \tag{3.8}$$

Solution. Applying Sumudu transform on both sides of (3.7), we get

$$S[(D_{0+}^{l-\frac{1}{2}} y)(t)](u) - \lambda S[y(t)](u) = 0. \tag{3.9}$$

Using (2.7),

$$u^{-(l-\frac{1}{2})} S[y(t)](u) - \sum_{j=1}^{l} d_j u^{-j} - \lambda S[y(t)](u) = 0$$

$$(u^{-(l-\frac{1}{2})} - \lambda)[Sy](u) = \sum_{j=1}^{l} d_j u^{-j}.$$

Using (3.2),

$$[Sy](u) = \sum_{j=1}^{l} d_j \frac{u^{l-j-\frac{1}{2}}}{(1 - \lambda u^{l-\frac{1}{2}})}. \tag{3.10}$$

Replacing $\alpha = (l - \frac{1}{2})$ and $\beta = l - j + \frac{1}{2}$ in (2.12) and taking inverse Sumudu transform on both sides of (3.10), The solution of (3.7) as

$$y(t) = \sum_{j=1}^{l} d_j y_j(t),$$

where

$$y_j(t) = t^{l-j-\frac{1}{2}} E_{l-\frac{1}{2},l-j+\frac{1}{2}}(\lambda t^{l-\frac{1}{2}}), j = 1, 2, ..., l.$$

4 Conclusion

We have obtained the solutions of homogeneous Liouville fractional differential equations with constant coefficients interms of Mittag-Leffler by Sumudu transform method. The results obtained provided fundamental system of solutions of the considered problem. Results obtained are validated with some examples.

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