Original Research Article

Investigation Accuracy of Crank Nicolson Methods and Its Modifications Scheme for One-Dimensional Linear convection-reaction-Diffusion Equations

Abstract

In this paper, Crank-Nicolson and Its Modification scheme are applied to find the solution of the convection-reaction-diffusion equation. First, the given solution domain is discretized. Then, using Taylor series expansion, we obtain the difference scheme of the model problem. By rearranging this scheme, we gain proposed techniques. To verify the validity of the proposed techniques, two model illustrations are considered. The stability and convergent analysis of the present scheme is worked by supporting the theoretical and numerical error bound. The accuracy of the present scheme has been shown in the sense of average absolute error, root means square error, and maximum absolute error norms. Then, the accuracy of the present techniques is compared with the accuracy obtained by another method in the literature. The physical behavior of the present results also has been shown in terms of graphs. As we can see from the results in tables and graphs, the present scheme approximates the exact solution veritably well and it's a relatively effective technique for solving convection-reaction-diffusion equation.

Keywords: Convection-reaction-diffusion equations, Crank Nicolsen scheme, Modifed Crank Nicolsen scheme, Stability and Convergent Analysis

1. Introduction

The application of PDEs can be planted in drugs, engineering, mathematics, and finance. Exemplifications include modeling mechanical vibration, heat, sound vibration, pliantness, and fluid dynamics. Indeed PDEs have a wide range of operations for real-world problems in wisdom and engineering [12]. Applications of advection-reaction-diffusion equations have been playing a significant role in numerous engineering applications [2]. The application of convection-reaction-diffusion is also found in transport models which are a necessary tool for describing colorful natural and anthropogenic processes [3, 7]. These model's problems contain parameters that must be specified to uniquely determine the result of boundary value problems. But in practice, situations frequently arise when some of these parameters are unknown or given roughly and they need to be determined or clarified [10, 11]. Applications of ecology, geophysics, and seismology have also arisen in the form of advection-reaction-diffusion or convection-reaction-diffusion equations. Depending on the equation under consideration and its boundary conditions, powerful techniques can be required to find its solution. The parameter associated with the model problem is used to govern the diffuseness/convection or transportation of particles from the high to low absorption portion of the particle. For example, the energy can be converted inside a physical system due to the convection and diffusion processes. This process is governed by the convection (diffusion) constant. The majority of PDEs don't have analytical solutions. One of these PDEs is a convection-diffusion equation. This convection-diffusion equation is a veritably important tool in the modeling of electrostatic systems and unfortunately may not only be solved analytically. Because analytical techniques are grounded in advanced fine ways.

Accordingly, numerical simulation must be employed for the PDEs model problem. Several numerical approaches are available to solve the convection-diffusion equation. The same numerical techniques are in general, simple but induce inaccurate results. Therefore, to solve this equation, these numerical techniques must be accruable. For instance, finite-difference techniques, Finite element, and boundary element techniques are traditional numerical techniques that we use to solve the convection-diffusion equation, see [12, 14]. The finite difference is a numerical technique, used to convert the convection-diffusion equation into their independently tried counterpart of the difference scheme. This system of difference equations may be readily solved by using matrix inversion, Jacobi, Gauss elimination, and the consecutive over-relaxation approach techniques. The accuracy of such a technique is thus directly tied to the competency of a finite grid to compare other numerical techniques. Hence, the accuracy of these methods may be arbitrarily increased by simply adding the number of samples/ grid points. It takes time and requires high storage capacity. Some other numerical techniques are based on finite-difference approximations [16, 19], integral transform methods [13, 20], Monte Carlo simulation [1, 9], variation iteration methods [21], and meshless method using radial basis functions [18]. Also, these techniques have high costs regarding magazine capacity and long computational time.

However, other accruable numerical techniques have been required to solve the convection-diffusion equations [4, 17]. Recently, different researchers have presented the numerical solution of the advection-diffusion or convection-reaction equation. Wang et.al. [2] Presented the numerical solution of the advection-diffusion equation by using radial cornerstone functions. They applied the Gaussian radial cornerstone functions for carrying the result of the direct advection-diffusion equation. The result diverges until the optimal shape parameters (Péclet number) are attained for the bases function. Hence the accumulation of the error generated within the attained approximate result. As compared to the exact result, the approximate result needs additional enhancement. To reduce the accumulation of these errors, the numerical algorithm must be monumentally accurate. To negotiate this thing, implicit schemes have been developed to solve this model problem. Therefore, this paper aims to construct an effective and accurate implicit numerical technique to find the outcome of one-dimensional convection-reaction-diffusion and advection-diffusion equation given in the form of:

$$C_t + \alpha C_x - \beta C_{xx} = f(x,t), (x,t) \in (0,1) \times (0,\ T) \qquad (1)$$

Subjected to initial and boundary conditions respectively given by:

$$C(x,0) = C_0(x) \quad 0 \leq x \leq 1$$

$$C(0,t) = g_0(t).\ C(1,t) = g_1(t) \quad , t \geq 0 \qquad (2)$$

Discretized the solution domain

Now we define a mesh size h and k. The discretization of the solution domain and induction of a grid by using invariant discretize of grid point given by

$$0 = x_0 < x_1 < x_2 < \cdots < x_M = 1,\ x_{j+1} = x_j + jh \ , h = \frac{1}{M};$$

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T\ , t_{n+1} = t_n + nk \ , k = \frac{T}{N}. \qquad (3)$$

where $j = 0(1)M$,$n = 0(1)N$. M and N are the number of grid points apart in the x and t- and directions. Additionally, the present paper is organized as follows. Section two is the Formulation of numerical techniques, Section three is Stability and convergency analysis, and Section four is the Results of numerical experimentations. Section five is the Discussion and Section six is the conclusion.

2. Formulation of the Numerical Techniques

2.1. Crank-Nicolson Scheme

Assuming that $C(x,t)$ has continuous higher order partial derivative on the region $D = [0,1]x[0,T]$. For the sake of simplicity, assume that $C(x_j,t_n) = C_{jn}$, $\frac{\partial^p C(x_j,t_n)}{\partial x^p} = \partial_x^p C_{j\,n}$ and $\frac{\partial^p C(x_j,t_n)}{\partial t^p} = \partial_t^p C_{j\,n}$ for $p \geq 1$ is p^{th} order derivatives. Then, the Taylor series expansions of $C(x_{j+1},t_n)$, $C(x_{j-1},t_n)$, $C(x_j,t_{n+1})$ and $C(x_j,t_{n-1})$ are given by:

$$C(x_{j+1},t_n) = C(x_j,t_n) + h\partial C(x_j,t_n) + \frac{h^2}{2!}\,\partial_x^2 C(x_j,t_n) + \frac{h^3}{3!}\partial_x^3 C(x_j,t_n) + \cdots \quad (4)$$

$$C(x_{j-1},t_n) = C(x_j,t_n) - h\partial C(x_j,t_n) + \frac{h^2}{2!}\,\partial_x^2 C(x_j,t_n) - \frac{h^3}{3!}\partial_x^3 C(x_j,t_n) + \cdots \quad (5)$$

$$C(x_j,t_{n+1}) = C(x_j,t_n) + k\partial C(x_j,t_n) + \frac{k^2}{2!}\,\partial_t^2 C(x_j,t_n) + \frac{k^3}{3!}\partial_t^3 C(x_j,t_n) + \cdots \quad (6)$$

$$C(x_j,t_{n-1}) = C(x_j,t_n) - k\partial C(x_j,t_n) + \frac{k^2}{2!}\,\partial_t^2 C(x_j,t_n) - \frac{k^3}{3!}\partial_t^3 C(x_j,t_n) + \cdots \quad (7)$$

By subtraction (5) from (4) and (7) from (6), the first-order central finite difference scheme of first-order partial derivatives of the given model problem in both directions:

$$\delta_{cx}^{(1)} C(x_j,t_n) = \frac{C_{j+1\,n} - C_{j-1n}}{2h} + \tau_1 \ , \ \delta_{ct}^{(1)} C(x_j,t_n) = \frac{C_{j\,n+1} - C_{j\,n-1}}{2k} + \tau_2 \quad (8)$$

Where respective $\tau_1 = -\frac{h^2}{6}\partial_x^3 C(x_j,t_n)$ and $\tau_2 = -\frac{k^2}{6}\partial_t^3 C(x_j,t_n)$ are their principal local truncation error terms. Again by adding (4) to (5), the second-order central finite difference scheme of the second-order partial derivative of the mode problem is:

$$\delta_{cx}^{(2)} C(x_j,t_n) = \frac{C_{j+1\,n} - 2C_{j\,n} + C_{j-1n}}{h^2} + \tau_3 \quad (9)$$

Where $\tau_3 = -\frac{h^2}{12}\partial_x^4 C(x_j,t_n)$ is their principal local truncation error term. Substitute (8) and (9) into (1) we obtain the difference scheme of the form of:

$$\frac{C_{j\,n+1} - C_{j\,n-1}}{2k} + \tau_2 + \alpha\left(\frac{C_{j+1\,n} - C_{j-1n}}{2h} + \tau_1\right) - \beta\left(\frac{C_{j+1\,n} - 2C_{j\,n} + C_{j-1n}}{h^2} + \tau_3\right) = f_{j\,n}$$

$$C_{j\,n+1} - C_{j\,n-1} + \theta(C_{j+1\,n} - C_{j-1n}) - \gamma(C_{j+1\,n} - 2C_{j\,n} + C_{j-1n}) = U_{jn} \quad (10)$$

where $\theta = \frac{\alpha k}{h}, \gamma = \frac{\beta k}{h^2}, U_{jn} = kf(x_j\,,t_n)$ and $\tau_{j\,n} = 2k\left(\beta\frac{h^2}{12}\partial_x^4 - \frac{k^2}{6}\partial_t^3 - \alpha\frac{h^2}{6}\partial_x^3\right)C(x_j,t_n)$ is a local truncation error. The above finite difference scheme is a central finite difference scheme and it is explicit method is simple but it has one serious drawback. Now we want to use the implicit method which is called the Crank-Nicolson method. Thus from the above scheme, we introduce the crank-Nicolson types of the numerical scheme given by:

$$C_{j\,n+1} - C_{j\,n-1} + \frac{\theta}{2}(C_{j+1\,n} - C_{j-1n} + C_{j+1\,n+1} - C_{j-1n+1})$$

$$-\frac{\gamma}{2}(C_{j+1\,n} - 2C_{j\,n} + C_{j-1n} + C_{j+1\,n+1} - 2C_{j\,n+1} + C_{j-1n+1}) = 2U_{jn}$$

$$2C_{j\,n+1} - 2C_{j\,n-1} + \theta\left(C_{j+1\,n} - C_{j-1n} + C_{j+1\,n+1} - C_{j-1n+1}\right)$$

$$-\gamma\left(C_{j+1\,n} - 2C_{j\,n} + C_{j-1n} + C_{j+1\,n+1} - 2C_{j\,n+1} + C_{j-1n+1}\right) = 4U_{jn}$$

$$(\theta+\gamma)C_{j+1\,n+1} + (2+2\gamma)C_{j\,n+1} + (\theta+\gamma)C_{j+1\,n+1} =$$

$$(\theta+\gamma)C_{j+1\,n} + (2+2\gamma)C_{j\,n} + (\theta+\gamma)C_{j-1\,n} - 4U_{jn} \qquad (11)$$

Now by introducing the vector 'C' such that $C_{n+1} = [C_{1n+1}, C_{2n+1}, C_{3n+1}, \dots, C_{M-1n+1}]^t$ and $C_n = [C_{1n}, C_{2n}, C_{3n}, \dots, C_{M-1n}]^t$ respectively into the left and right-hand side of the system of differential equations in (11) we can rewrite it as matrix form which is given by:

$$AC_{n+1} = BC_{n+1} + d_n \qquad (12)$$

where $$A = \begin{pmatrix} 2+2\gamma & \theta+\gamma & 0 & & 0 & 0 \\ \theta+\gamma & 2+2\gamma & \theta+\gamma & \cdots & 0 & 0 \\ 0 & \theta+\gamma & 2+2\gamma & & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 2+2\gamma & \theta+\gamma \\ 0 & 0 & 0 & & \theta+\gamma & 2+2\gamma \end{pmatrix}$$

$$B = \begin{pmatrix} 2+2\gamma & \theta+\gamma & 0 & & 0 & 0 \\ \theta+\gamma & 2+2\gamma & \theta+\gamma & \cdots & 0 & 0 \\ 0 & \theta+\gamma & 2+2\gamma & & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 2+2\gamma & \theta+\gamma \\ 0 & 0 & 0 & & \theta+\gamma & 2+2\gamma \end{pmatrix}$$

$$d_{1,n} = (\theta+\gamma)C_{0\,n+1} + (\theta+\gamma)C_{0\,n} + 2U_{jn}\,,\ d_{j,n} = 2U_{jn}, j = 1(1)M-1$$

$$d_{M,n} = (\theta+\gamma)C_{M\,n+1} + (\theta+\gamma)C_{M\,n} + 2U_{jn}\,.$$

2.2. Modified crank-Nicolson scheme

The Modified Crank-Nicolson scheme obtained by Kweyu et al. [22] and used by Feyisa and Hailu [15] combined with Richardson extrapolation to solve the classical one-dimensional heat equation is an accruable and stable implicitly numerical technique. Based on this work, we develop the modified crank-Nicolson scheme for the one-dimensional linear convection-reaction-diffusion equation given in (1). Let us rewrite the model problem in (1) in the form of:

$$\delta_{ct}^{(1)}C_{j\,n} + \frac{\theta}{3}\delta_{cx}^{(1)}\left(C_{j\,n-1} + C_{j\,n} + C_{j\,n-1}\right) - \frac{\theta}{3}\delta_{cx}^{(2)}\left(C_{j\,n-1} + C_{j\,n} + C_{j\,n+1}\right) = f(x_{j\,,}t_n) \quad (13)$$

By using (8), (9), and (13), we obtain:

$$C_{j\,n+1} - C_{j\,n-1} + \frac{2\theta}{3}\left(C_{j+1\,n-1} - C_{j-1n-1} + C_{j+1\,n} - C_{j-1n} + C_{j+1\,n+1} - C_{j-1n+1}\right)$$

$$-\frac{2\gamma}{3}\left(C_{j+1\,n-1} - 2C_{j\,n-1} + C_{j-1n-1} + C_{j+1\,n} - 2C_{j\,n} + C_{j-1n} + C_{j+1\,n+1} - 2C_{j\,n+1} + C_{j-1n+1}\right) = 4U_{jn}$$

$$3C_{j\,n+1} - 3C_{j\,n-1} + 2\theta\left(C_{j+1\,n-1} - C_{j-1n-1} + C_{j+1\,n} - C_{j-1n} + C_{j+1\,n+1} - C_{j-1n+1}\right)$$

$$-2\gamma\left(C_{j+1\,n-1} - 2C_{j\,n-1} + C_{j-1n-1} + C_{j+1\,n} - 2C_{j\,n} + C_{j-1n} + C_{j+1\,n+1} - 2C_{j\,n+1} + C_{j-1n+1}\right) = 12U_{jn}$$

$$2(\theta+\gamma)C_{j+1\,n+1} + (3+4\gamma)C_{j\,n+1} + 2(\theta+\gamma)C_{j+1\,n+1} =$$

$$2(\theta+\gamma)C_{j+1\,n} + (3+4\gamma)C_{j\,n} + (\theta+\gamma)C_{j-1\,n} +$$

$$-2(\theta+\gamma)C_{j+1\,n-1} + (4\gamma+3)C_{j\,n-1} - 2(\gamma-\theta)C_{j-1\,n-1} - 12U_{jn} \tag{14}$$

By introducing the vector form of the unknown variable, the matrix form of (14) is given by:

$$DC_{n+1} = EC_{n+1} + g_n \tag{15}$$

where

$$D = \begin{pmatrix} 3+4\gamma & 2(\theta+\gamma) & 0 & & 0 & 0 \\ \theta+\gamma & 3+4\gamma & 2(\theta+\gamma) & \cdots & 0 & 0 \\ 0 & 3(\theta+\gamma) & 3+4\gamma & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 3+4\gamma & 2(\theta+\gamma) \\ 0 & 0 & 0 & & 2(\theta+\gamma) & 3+4\gamma \end{pmatrix}$$

$$E = \begin{pmatrix} 3+4\gamma & 2(\theta+\gamma) & 0 & & 0 & 0 \\ 2(\theta+\gamma) & 3+4\gamma & 2(\theta+\gamma) & \cdots & 0 & 0 \\ 0 & 2(\theta+\gamma) & 3+4\gamma & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 3+4\gamma & 2(\theta+\gamma) \\ 0 & 0 & 0 & & 2(\theta+\gamma) & 3+4\gamma \end{pmatrix}$$

$g_{1,n} = 2(\theta+\gamma)C_{0\,n+1} + 2(\theta+\gamma)C_{0\,n} + 2(\theta+\gamma)C_{2\,n-1} + (3+4\gamma)C_{1\,n-1} - 2(\theta-\gamma)C_{-1\,n-1} +$
$12U_{1n}$

$g_{j,n} = 12U_{jn} - 2(\theta+\gamma)C_{j+1\,n-1} + (3+4\gamma)C_{j\,n-1} - 2(\theta-\gamma)C_{j-1\,n-1}, j = 1(1)M-1$

$g_{M,n} =$
$2(\theta+\gamma)C_{M\,n+1} + 2(\theta+\gamma)C_{M\,n} - 2(\theta+\gamma)C_{M\,n-1} + (3+4\gamma)C_{M-1\,n-1} - 2(\theta-\gamma)C_{M-2\,n-1}U_{jn}$

and for $U_{jn} = 12\Delta t f(x_j, t_n)$. Then write the MATLAB code for the scheme in (12) and (15) by using the matrix inverse method, we can find the solution of the given model problems.

3. Stability and Convergent Analysis for Proposed Methods

The Von-Neumann stability analysis technique is applied to investigate the stability of the proposed method. Such an approach has been used by many researchers like,[5, 23, 24, 25, 26]. Now assume that the solution to the given problem at the point $(x_j, t_n\ )$ is

$$u_{j\,n} = \lambda^n e^{ijhK_p} \tag{17}$$

Where $i = \sqrt{-1}$,$K_p = \frac{p\pi}{M}$ is real and $\lambda \in \mathbb{C}$ complex number and $p = 1(1)M$. Substituting Eq. (17) into Eq. (11) we obtain:

$$(\theta+\gamma)\lambda^{n+1}e^{ihK_p(j+1)} + (2+2\gamma)\lambda^{n+1}e^{ijhK_p} + (\theta+\gamma)\lambda^{n+1}e^{ihK_p(j-1)} =$$

$$(\theta+\gamma)\lambda_p^n e^{ihK_p(j+1)} + (2+2\gamma)\lambda^n e^{ijhK_p} + (\theta+\gamma)\lambda^n e^{ihK_p(j-1)} - 4\Delta t\lambda^n e^{ijhK_p} \qquad \text{(i)}$$

Dividing both side equation (i) sides by $\lambda^n e^{ijhK_p}$, we obtain:

$$\left((\theta+\gamma)e^{ihK_p} + (2+2\gamma) + (\theta+\gamma)e^{-ihK_p}\right)\lambda_p =$$

$$(\theta+\gamma)e^{ihK_p} + (2+2\gamma) + (\theta+\gamma)e^{-ihK_p} - 4\Delta t$$

$$\left[2(\theta+\gamma)\cos(hK_p) + (2+2\gamma)\right]\lambda_p = 2(\theta+\gamma)\cos(hK_p) + (2+2\gamma) - 4\Delta t$$

$$\lambda_p = \frac{[(\theta+\gamma)\cos(hK_p)+(1+\gamma)]-2\Delta t}{[(\theta+\gamma)\cos(hK_p)+(1+\gamma)]} \tag{18}$$

Since $\Delta t \ll 1$ and $2\Delta t \ll 1$. Therefore for any h,$|\cos(hK_p)| \leq 1$ and $[(\theta+\gamma)\cos(hK_p) + (1+\gamma)] - 2\Delta t \ll 1$. But$[(\theta+\gamma)\cos(hK_p) + (1+\gamma)] - 2\Delta t \ll [(\theta+\gamma)\cos(hK_p) + (1+\gamma)]$. This implies $\frac{[(\theta+\gamma)\cos(hK_p)+(1+\gamma)]-2\Delta t}{[(\theta+\gamma)\cos(hK_p)+(1+\gamma)]} \ll 1$. Thus we obtain the stability criteria such that $max|\lambda_p| < 1$ for all $p = 1(1)M$.Therefore the proposed cark-Nicolson scheme is stable. Now again to investigate the stability of the modified scheme, substituting equation (17) into (14) we obtain:

$$2(\theta+\gamma)\lambda^{n+1}e^{ihK_p(j+1)} + (3+4\gamma)\lambda^{n+1}e^{ijhK_p} + 2(\theta+\gamma)\lambda^{n+1}e^{ihK_p(j-1)} =$$

$$2(\theta+\gamma)\lambda_p^n e^{ihK_p(j+1)} + (3+4\gamma)\lambda^n e^{ijhK_p} + 2(\theta+\gamma)\lambda^n e^{ihK_p(j-1)}$$

$$2(\theta+\gamma)\lambda_p^{n-1}e^{ihK_p(j+1)} + (3+4\gamma)\lambda^{n-1}e^{ijhK_p} + 2(\theta+\gamma)\lambda^{n-1}e^{ihK_p(j-1)} - 12\Delta t\lambda^n e^{ijhK_p} \quad \text{(ii)}$$

Dividing both sides of equation (ii) by $\lambda^n e^{ijhK_p}$, we obtain:

$$\left(2(\theta+\gamma)e^{ihK_p} + (3+4\gamma) + 2(\theta+\gamma)e^{-ihK_p}\right)\lambda_p =$$

$$2(\theta+\gamma)e^{ihK_p} + 2(2+2\gamma) + 2(\theta+\gamma)e^{-ihK_p} - 12\Delta t$$

$$\left(-2(\theta+\gamma)e^{ihK_p} + (4\gamma+3) - 2(\gamma-\theta)e^{-ihK_p}\right)\lambda_p^{-1}$$

$$\left[4(\theta+\gamma)\cos(hK_p) + (3+4\gamma)\right]\lambda_p = 4(\theta+\gamma)\cos(hK_p) + (2+2\gamma) - 12\Delta t$$

$$\left[-2(\theta+\gamma)\left(\cos(hk_p) + isin(hk_p)\right) - 2(\gamma-\theta)\left(\cos(hk_p) - isin(hk_p)\right) + (4\gamma+3)\right]\lambda_p^{-1}$$

$$\left[4(\theta+\gamma)\cos(hK_p) + (3+4\gamma)\right]\lambda_p = 4(\theta+\gamma)\cos(hK_p) + (2+2\gamma) - 12\Delta t$$

$$-4\left[\gamma\cos(hk_p) + isin(hk_p)\right]\lambda_p^{-1}$$

$$\left[4(\theta+\gamma)\cos(hK_p) + (3+4\gamma)\right]\lambda_p^2 - (4(\theta+\gamma)\cos(hK_p) + (2+2\gamma) - 12\Delta t)\lambda_p$$

$$=-4\left[\gamma\cos(hk_p) + isin(hk_p)\right]$$

$$\lambda_p^2 - \frac{(4(\theta+\gamma)\cos(hK_p) + (2+2\gamma) - 12\Delta t)}{4(\theta+\gamma)\cos(hK_p) + (3+4\gamma)}\lambda_p = -\frac{4\left[\gamma\cos(hk_p) + isin(hk_p)\right]}{4(\theta+\gamma)\cos(hK_p) + (3+4\gamma)}$$

Letting $X = (4(\theta+\gamma)\cos(hK_p) + (2+2\gamma) - 12\Delta t), Y = 4(\theta+\gamma)\cos(hK_p) + (3+4\gamma)$ and $Z = 4\left[\gamma\cos(hk_p) + isin(hk_p)\right]$. By using a perfect square from the above equation, we obtain:

$$\lambda_p^2 - \frac{X}{Y}\lambda_p = -\frac{Z}{Y} \Leftrightarrow \left(\lambda_p - \frac{X}{2Y}\right)^2 = \frac{1}{4}\left(\frac{X}{Y}\right)^2 - \frac{Z}{Y}$$

$$\lambda_p = \frac{X}{2Y} \pm \sqrt{\frac{1}{4}\left(\frac{X}{Y}\right)^2 - \frac{Z}{Y}}$$

$$\lambda_p = \frac{X}{2Y} \pm \frac{1}{2Y}\sqrt{X^2 - 4YZ} = \frac{1}{2Y}\left[X \pm \sqrt{X^2 - 4YZ}\,\right] \quad (19)$$

$\Delta t \ll 1$ Implies that $12\Delta t \le 1$. Hence for any $h, -1 < \cos(hK_p), sin(hk_p) < 1$. This implies that $\left|= (4(\theta+\gamma)\cos(hK_p) + (2+2\gamma) - 12\Delta t)\right| \ll 1$ and $\left|4\left[\gamma\cos(hk_p) + isin(hk_p)\right]\right| \ll 1$.

But $\left|4(\theta+\gamma)\cos\left(hK_p\right)+(3+4\gamma)\right| \gg 1$. Hence by using triangular inequality from equation (19) we obtain:

$$|\lambda_p| = \left|\frac{X}{2Y} \pm \frac{1}{2Y}\sqrt{X^2-4YZ}\right| < \frac{1}{2}\left|\frac{1}{Y}\right|\left[|X| + \sqrt{|X^2|-4|Y||Z|}\,\right] \ll 1$$

This is the same as true with the previous condition. It means $\max_{1\leq p\leq M}|\lambda_p| < 1$

So the developed Modified crank-Nicolson scheme for solving the given advection-reaction-convection-diffusion equation is also stable.

Theorem 2: The difference equation given in (11) and (14) are stable if real parts of eigenvalues of coefficient matrixes of the system of the obtained differential equation are satisfied $Real\left(\lambda_j\right) < 0$.

Proof: See (ref. 5)

Definition 1: Matric form of linear difference discretization scheme given in (11) and (14) are stable if, for sufficiently small mesh size h and k, there is a constant "G" independent of mesh size h and k such that

$$\left|\epsilon_{j\,n}\right| \leq G\left\{\max\left(|\epsilon_{0\,0}|, |\epsilon_{M\,N}|\right) + \max_{\substack{1\leq j\leq M\\ 1\leq n\leq N}}\left|T_{j\,n}\right|\right\} \tag{20}$$

where $\epsilon_{j\,n}$ is any global error function of the discretization scheme.

Theorem 2: The difference scheme defined in (11) and (14) is stable for any constant G independent of the mesh size.

Proof: Let $\epsilon_{.n} = u(x_., t_n) - u_{\,n}$ is the global error of solution of a given problem in Eq. (1) obtained by the proposed method, at the grid point $(x_., t_n)$, where $u(x_., t_n)$ and $u_{\,n}$ are exact and numerical solutions respectively. The global error in numerical calculation is the sum of round-off and truncation errors that exist in computation. Substitute Eq. (20) into (11) and (14) and using triangular inequality of matrix norm, we obtain:

$$|\epsilon_{n+1}| < G[\max\left(|\epsilon_0|, |\epsilon_N|\right) + \max_{1\leq n\leq N}|\tau_n|]$$

$$|\varepsilon_{n+1}| < H[\max\left(|\varepsilon_0|, |\epsilon_N|\right) + \max_{1\leq n\leq N}|\tau_n|]$$

Where $G = ||B||/||A||$ and $H = ||E||/||D||$ respectively. Hence, all of our proposed schemes are stable. Since from all principal parts of the derived local truncation error of the proposed scheme, we have:

$$\tau_{j\,n} = 2kh^2 \left(\beta \frac{1}{12} \partial_x^4 - \frac{1}{6h^2} \partial_t^3 - \alpha \frac{1}{6} \partial_x^3\right) C(x_j, t_n)$$

Thus, $\tau_{j\,n} \to 0$ as $h, k \to 0$. Therefore, our numerical schemes are consistent. Hence the schemes are convergent. Thus, the average absolute error (AAE), root mean square error (L_2) and maximum absolute error (L_∞) norms are used to measure the accuracy of the proposed method. The root means square error and maximum absolute error norms are calculated as in [2] given by.

$$L_2 = \sqrt{\frac{1}{N}\sum_{j=1}^{N} \left|C(x_j\,, t_n) - C_{j\,n}\right|^2}\,, L_\infty = \max_{1\le j\le M} \left|C(x_j\,, t_N) - C_{j\,N}\right|$$

An average absolute error norm (L_1) is also calculated by:

$$\text{AAE} = \frac{1}{N}\sum_{j=1}^{N} \left|C(x_j\,, t_n) - C_{j\,n}\right|$$

Where $C(x_j\,, t_n)$ and $C_{j\,n}$ are respectively exact and numerical solutions of the given model problem at the grid point $(x_j\,, t_n)$.

4. Results of Numerical Experiments

To test the validity of the proposed method, we have considered the following two model problems. Numerical results and errors are computed and the outcomes are represented tabularly and graphically.

Example1: Consider the advection-diffusion equation considered by Wang F. et.al. [2] given by:

$$C_t + \alpha C_x = \beta C_{xx} + \mathrm{g(x,t)}, 0 < x < 1, 0 < t < T,$$

Subjected to initial and boundary conditions respectively given by:

$$C(x,0) = ae^{bx}\ 0 \le x \le 1$$

$$C(0,t) = ae^{bt}, \qquad C(0,t) = ae^{bt}, 0 \le t \le T$$

And the exact solution is $C(x,t) = e^{bx+ct}$ and the values of parameters are $b = 1.25, \beta = 0.1, a = 0.1$, $\alpha = 0.1$, $c = \frac{\alpha \pm \sqrt{\alpha^2 + 4b\beta}}{2\beta}$, $\mathrm{g(x,t)} = 0$.

Example 2: Consider the convection-reaction-diffusion equation considered by Wang F. et.al. [2] given as:

$$C_t + \alpha C_x = \beta C_{xx} + \mathrm{g(x,t)}, 0 < x < \ 1{,}0 < t < T,$$

Subjected to initial and boundary conditions respectively given by:

$$C(x,0) = \sin(x) \ 0 \le x \le \ 1$$

$$C(0,t) = e^{-at}\sin(-bt), \qquad C(0,t) = e^{-at}\sin(1-bt), 0 \le t \le T$$

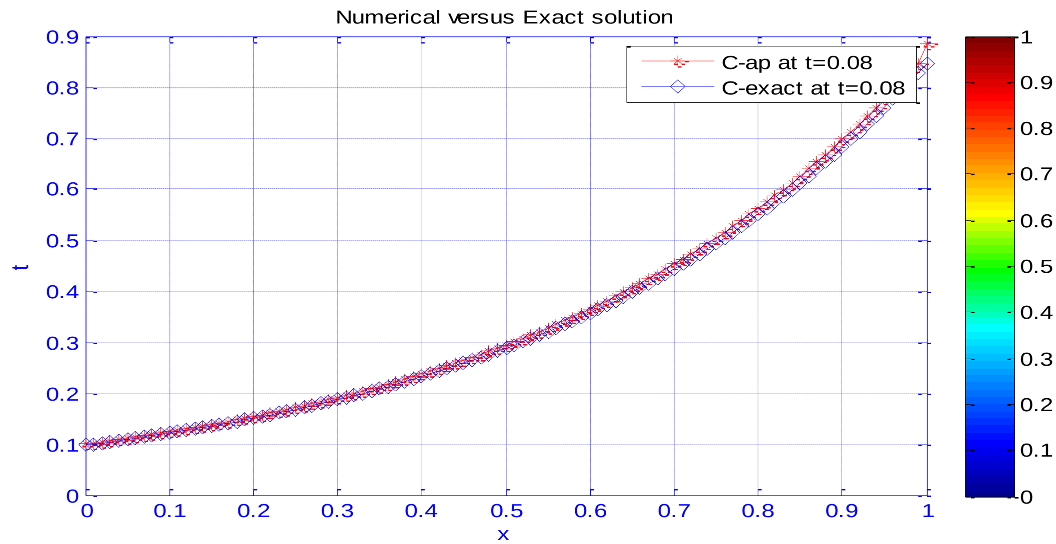
And its exact solution is $C(x,t) = e^{-at}\sin(x-bt)$ and the value of parameter are $b = 1$, β =1, a=1, α=1, $\mathrm{g(x,t)} = 0$

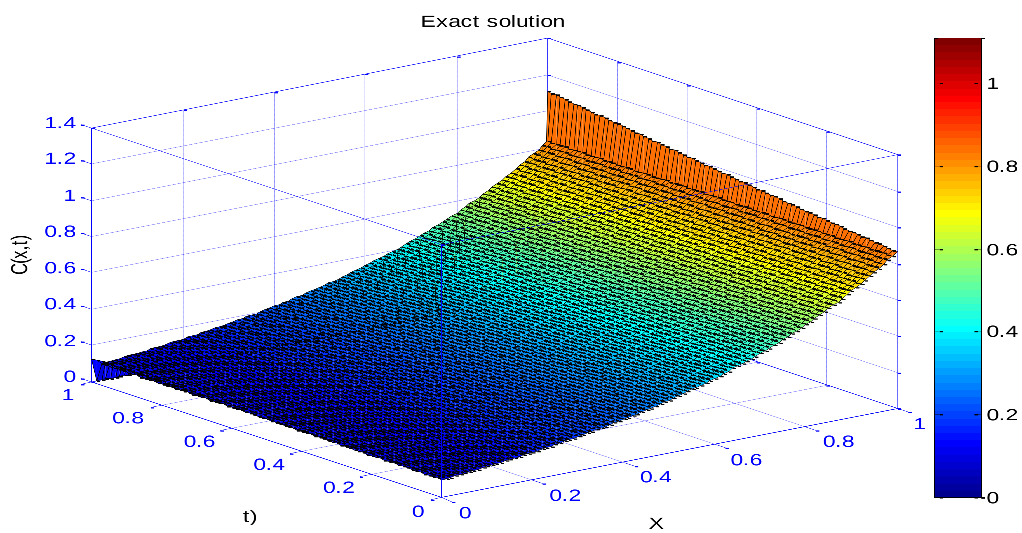
Table 1. Comparison of average absolute error for the problem given in example one in computations domain carried out until final time $T = 1$ with different mesh size h and time step k and at time value $t = 0.1$, $t = 0.5$, and $t = 1$.

Average Absolute error of Numerical Result obtained by Method in [27-29]			Average Absolute error of Numerical Result obtained by Method in [2]	Average Absolute error of Numerical Result obtained by present methods	
By-FD -scheme [28]	TPS-scheme [28]	CS-scheme [29]	By-DMM-scheme [2]	By Crank-Nicolson scheme	By Modified Crank-Nicolson scheme
AAE	AAE	AAE	AAE	AAE	AAE
1.0E-04	0.0000	3.0E-04	7.95E−06	3.3772E-06	2.1741E-06
8.0E-04	1.0E-04	1.0E-03	5.57E−06	8.5189E-07	5.2131E-07
1.8E-04	2.0E-04	2.3E-03	8.35E−6	3.8673E-07	2.1392E-07

Table 2. Comparison of Average Absolute error, Root Mean Square Error, and pointwise maximum absolute error between the exact solution and numerical solution obtained by the present method for the problem give an example one in computations domain carried out until final time $T = 1$ with different mesh size h and time step k .

The specified size of Grid Points in both direction		By Crank-Nicolson scheme			By Modified Crank-Nicolson scheme		
h	k	L_∞	L_2	AAE	L_∞	L_2	AAE
1/40	1/20	2.1674E-04	4.8464E-05	1.0837E-05	2.1674E-04	4.8464E-05	1.0837E-05
1/80	1/50	4.5774E-05	6.4734E-06	9.1548E-07	4.5774E-05	6.4734E-06	9.1548E-07
1/100	1/80	2.3166E-05	2.5900E-06	2.8957E-07	2.3166E-05	2.5900E-06	2.8957E-07
1/200	1/100	9.3878E-06	9.3878E-07	9.3878E-08	9.3878E-06	9.3878E-07	9.3878E-08





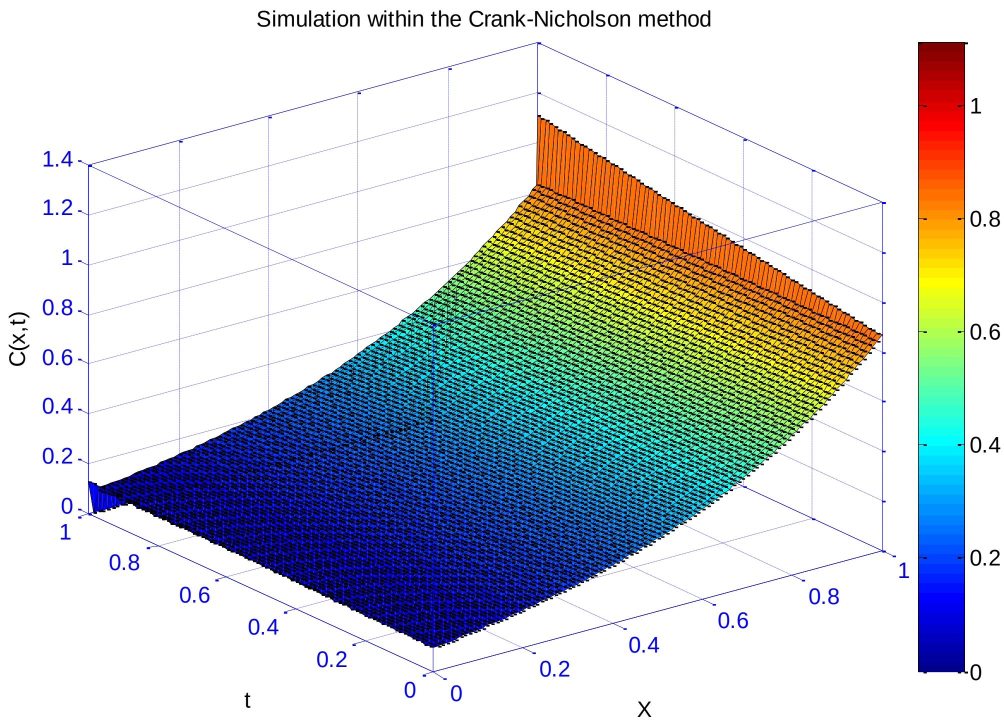
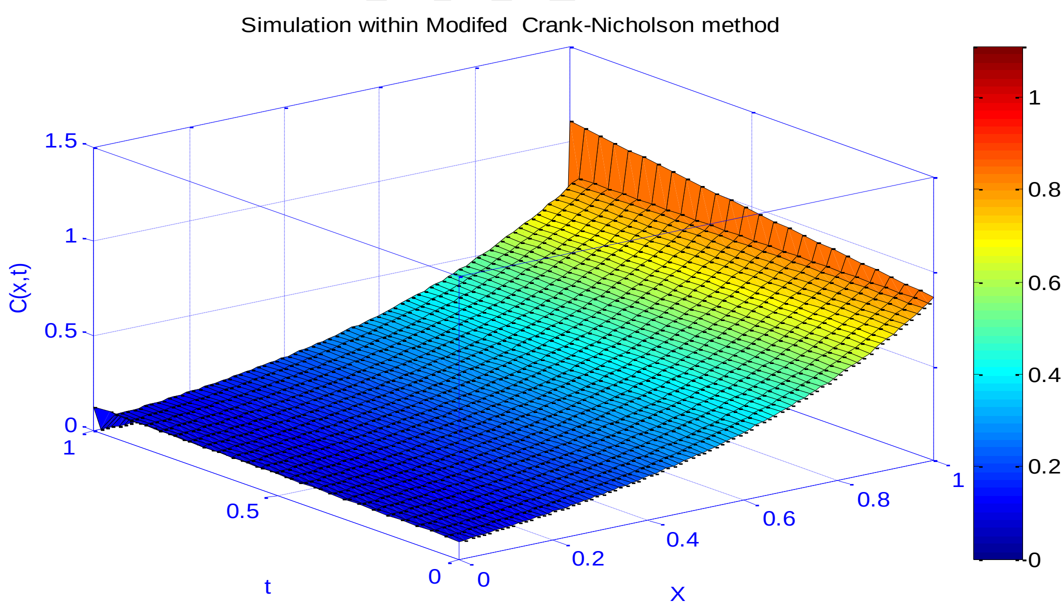


Figure 1: Physical behavior of Exact and Numerical Solution obtained by the crank-Nicolson method, for example, one on the uniform mesh of a maximum number of grid points $M = 100$ & time step $N = 80$.



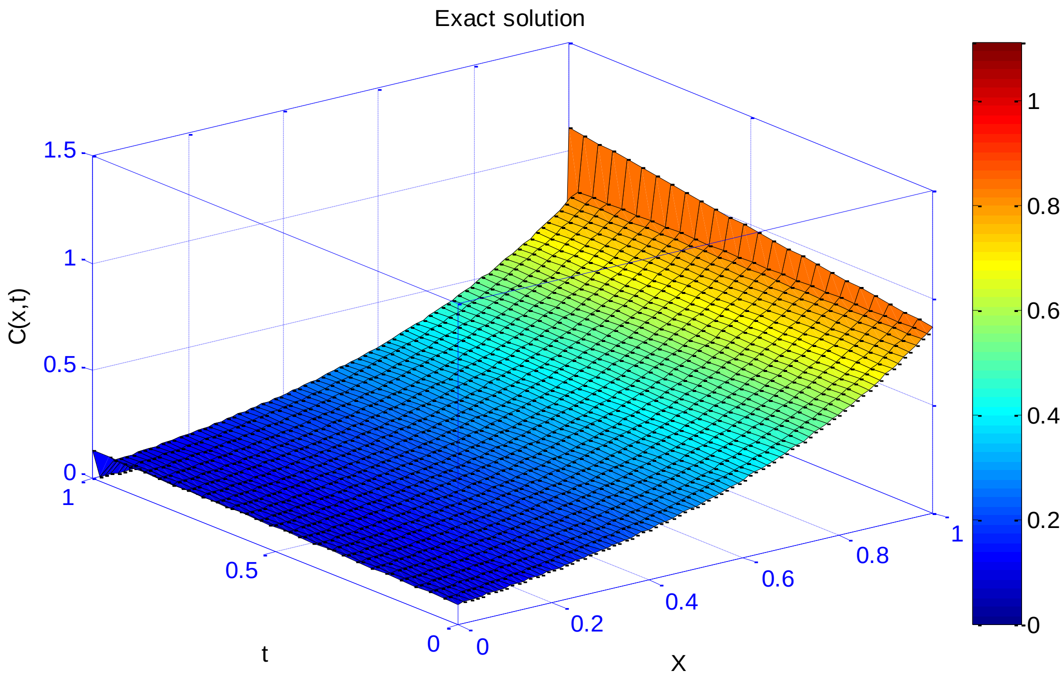


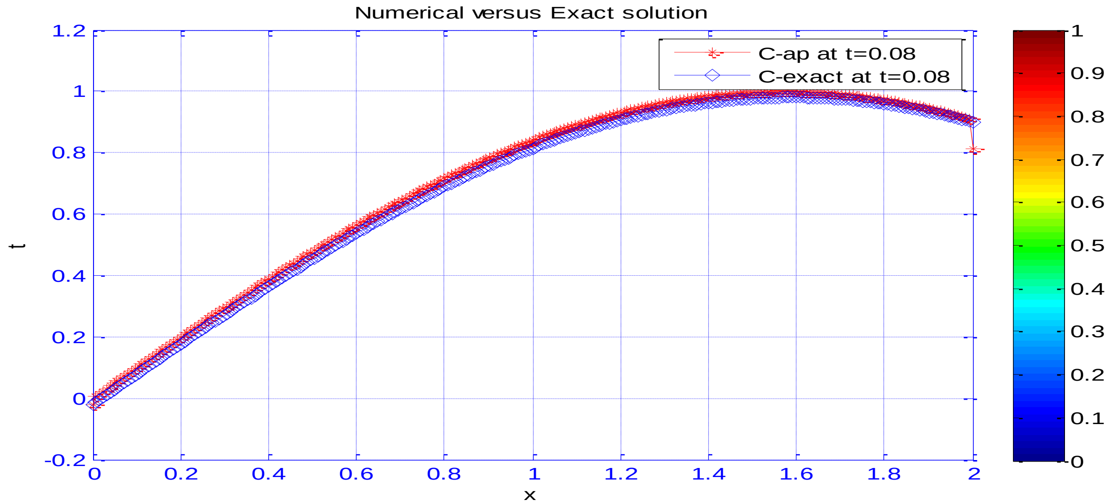
Figure 2: Physical behavior of Exact and Numerical Solution obtained by Modified crank-Nicolson method, for example, one on the uniform mesh of the maximum number of grid points $M = 80$ & time step $N = 25$.

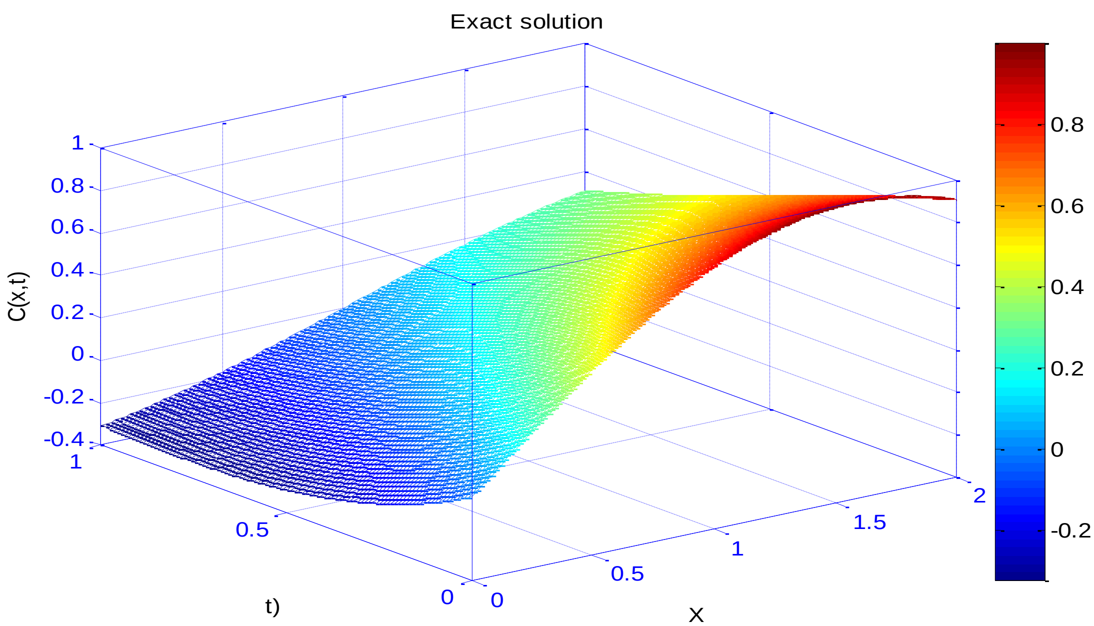
Table 3. Comparison of average absolute error for the problem given in example two in computations domain carried out until final time $T = 1$ with different mesh size h and time step k and at time value $t = 0.1,\quad t = 0.5,\quad$ and $t = 1$.

The specified size of grid points		Average Absolute error of Numerical Result obtained by Method in [2, 27]		Average Absolute error of Numerical Result obtained by present methods	
		By-FD -scheme[27]	By-DMM-scheme [2]	By Crank-Nicolson scheme	By Modified Crank-Nicolson scheme
h	k	AAE	AAE	AAE	AAE
1/80	1/100	1.4 E−06	9.3E−08	3.9795E-07	1.7034E-07
1/250	1/125	1.8 E−06	1.7 E−08	2.7651E-07	1.3560E-07
1/300	1/150	7.9 E−06	2.3 E−08	9.9666E-08	6.7128E-08

Table 4. Comparison of Average Absolute error, Root Mean Square Error, and pointwise maximum absolute error Between Exact solution and Numerical solution obtained by the present method for the problem given in example one in computations domain carried out until final time $T = 1$ with different mesh size h and time step k .

The specified size of Grid Points in both direction		By Crank-Nicolson scheme			By Modified Crank-Nicolson scheme		
h	k	L_∞	L_2	AAE_1	L_∞	L_2	AAE
1/10	1/10	1.5088E-03	3.3738E-04	7.5441E-05	3.1586E-03	7.0628E-04	1.5793E-05
1/30	1/25	1.2177E-03	2.4354E-04	4.8708E-05	2.0112E-04	2.5964E-05	3.3519E-06
1/50	1/45	6.6957E-04	1.2225E-04	2.2319E-05	1.2573E-04	1.4057E-05	1.5716E-06
1/70	1/65	3.4671E-04	4.9033E-05	6.9343E-06	1.0751E-04	1.2020E-05	1.3439E-06





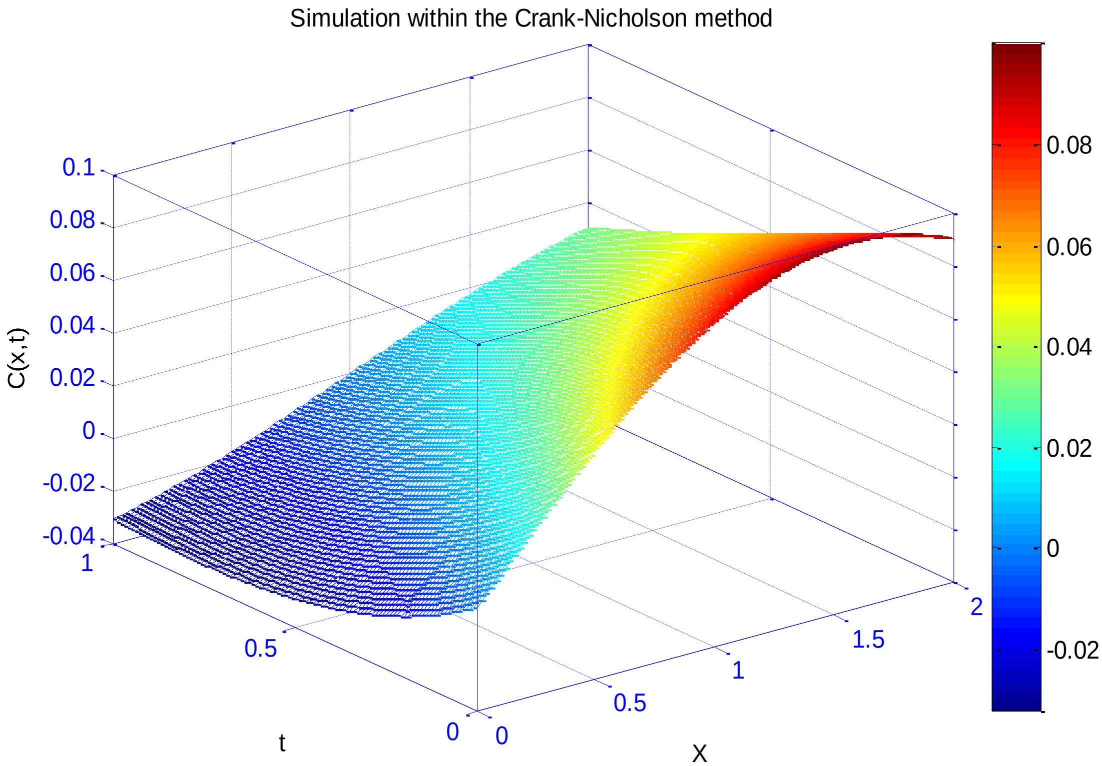


Figure 3: Physical behavior of Exact and Numerical Solution obtained by Modified crank-Nicolson method, for example two on the uniform mesh of the maximum number of grid points $\boldsymbol{M = 80}$ & time step $\boldsymbol{N = 25}$.

5. Discussion

In this paper, the Crank-Nicolson and Its Modification scheme are presented for solving a one-dimensional Advection-Convection-reaction equation. To demonstrate the competence of the present method, two model examples are solved by taking different values for step size h, and time step k. The computation of numerical results obtained by the present method has been presented in terms of average absolute error, root mean square error, and point-wise maximum absolute error calculated between exact and numerical solutions, and the results are summarized in Tables and graphs. Moreover, in the present numerical computation, the result presented in Tables 1 and 2 shows that average absolute error (AAE), roots mean square error (L_2) and pointwise maximum absolute error (L_∞) norms for the solution of example one are increasing as the number of mesh points Mx and Nt increases in both directions. But, the accuracy of the present methods is increasing, and it is superior to the accuracy of the method in [2, 27, 28, 29]. Figure 1 shows that, the physical background of continuity solution within the traditional form of a conservation law (The rate of change of the total amount of C in any sections of $0 < x < 1$ must be balanced by the net inflow across 0 to 1). Again the result presented in table 3 and 4 also shows that all average absolute error (AAE), root mean square error (L_2) and maximum absolute error (L_∞) norms for the solution of example two are increasing rapidly as step length and time step decrease (i.e., M and N are simultaneously increased) when the model problem is subjected to the periodic initial condition in $0 \leq x \leq 2$ and until the final time $= 1$. Also, in this case, the accuracy of the present method is rapidly increased. But when we compare the accuracy of the method in [2, 27] with the accuracy of present methods, the accuracy of present methods is superior to the accuracy of the method in [27]. However, the accuracy of the present method is agreed with the accuracy of the method in [2]. Further, as can be shown in Figure 2, the proposed method approximates the solution for example two very well for different values of step length h and time step k . The result presented in Tables 2 and 4 also shows that roots mean square error norm (L_2) and maximum absolute error norm (L_∞) are increasing rapidly as step length and time step decrease respectable for Examples one and two. Also, in this case, the accuracy of the present method is rapidly increased. But when we compare the present methods, the accuracy of these schemes is uniformly increasing as step length and time step decreases. In all cases, the solution specifies the behavior of error produced in the solution by the present method within the effects of mesh sizes in the solution domain. Further, to accomplish the accuracy of the present methods, both the theoretical and numerical error bounds have been established. The results in the Tables and graphs further confirmed the established computational rate of convergence and theoretical estimates of error bounds.

6. Conclusion

Approaches, the Crank-Nicolson, and Its modification scheme are used to solve convection-reaction-diffusion equations numerically. The comparison of the accuracy of the obtained numerical solution with the accuracy of a solution is presented in the table. From this table, we conclude that the present scheme is more convenient, reliable, and effective than this entire scheme. As can be seen in the present methods, the accuracy improves when the number of grid points is increasing. In summary, the present scheme is capable of solving convection-reaction equations.

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