Small Data Scattering for the Global Solutions of the Supercritical Generalized KdV Equation

Abstract. We consider the scattering problem for the global solutions of the supercritical generalized KdV equation $\partial_t u+\partial_{xxx}u+\mu\partial_x(u^{k+1}) = 0$, where $k > 4$ is an integer, initial data u_0 belongs to $H^1(\mathbb{R})$, and $\mu = \pm 1$.

To solve the scattering problem, a scattering criteria is established firstly, and then a new inequality is introduced to obtain uniformly bounded solutions in $H^1(\mathbb{R})$. Finally, we further clarify the conditions for the equation to have a global solution scattering in $H^1(\mathbb{R})$. Our method is mainly inspired by the works of Farah, Linares, Pastor, and Visciglia.

Keywords: supercritical; generalized KdV equation; scattering; global solution.

1 Introduction

It is well known that partial differential equations (PDES) are widely used to simulate natural phenomena. And many physical phenomena and laws can be described by appropriate PDES, such as the Schrödinger equation describing the basic laws of quantum mechanics, the KdV equation describing solitary waves. These equations have become important mathematical physics equations that have attracted wide attention and in-depth research. Here we mainly study the long-time asymptotic behavior of the solutions of nonlinear equations — scattering theory.

We say that the solution of the nonlinear equation scatters in space X , if the solution of the nonlinear equation approaches a solution of the associated linear equation in a certain norm sense (the initial values of these two equations can be different, but they both belong to the same space X), when t approaches infinity. From the definition, it can be seen that if we know that a solution of the nonlinear equation has scattering behavior, we can indirectly obtain the properties that the solution should satisfy, by studying the properties of the solution to the associated linear equation. Therefore, it is very important and meaningful to study the scattering of solutions of nonlinear equations.

In fact, there are many studies on the scattering theory of nonlinear equations. Take the nonlinear Schrodinger equation as an example. Kenig, Merle [1], using a concentration compactness method, proved the scattering for the energy-critical, focusing, nonlinear Schrödinger equation in the radial case. For the radial $H^1(\mathbb{R})$ solutions to the focusing cubic 3D nonlinear Schrödinger equation, Holmer, Roudenko [2] also obtained a sharp condition for scattering (the radial case). The technique employed is parallel to that in [1]. And the general case was finished by Duyckaerts et al [3]. Fang, Xie, Cazenave [4] extend those result to arbitrary space dimensions and focusing, mass-supercritical and energy-subcritical power nonlinearities. There are many other works related to the Schrödinger equation [5-7]. Moreover, nonlinear wave equations also have similar scattering theories [8,9]. In this paper, we mainly study the scattering theory for the generalized Korteweg-de Vries (gKdV) equation.

We consider solutions of the Initial Value Problem (IVP) associated with the following supercritical gKdV equation, i.e.,

$$\begin{cases} \partial_t u + \partial_{xxx} u + \mu \partial_x (u^{k+1}) = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $u_0 \in H^1(\mathbb{R})$, $k > 4$ is an integer, $\mu = \pm 1$. We say that the equation (1.1) is focusing if $\mu = 1$, and defocusing if $\mu = -1$. When $k \in \mathbb{Z}^+$, we call the above family of equations the k-generalized KdV equations (k-gKdV). For $k < 4$, the equation (1.1) is subcritical case, $k = 4$ is the critical case. And $k > 4$ is the supercritical case, here we mainly consider the supercritical case. The case $k = 1$ is the well-known KdV equation, which was first derived as a model for unidirectional propagation of nonlinear dispersive long waves. When $k = 2$, (1.1) is called the modified KdV (mKdV) equation. Like the KdV equation, it models propagation of weak nonlinear dispersive waves and it can also be solved by inverse scattering theory. The mKdV equation is another fundamental completely integrable system in solitary wave theory.

Moreover, both the KdV and the mKdV equation have an infinite number of conserved quantities [10], but when $k > 2$, the conclusion is not valid. However, all real-valued solutions of k-gKdV equations have the following two conserved quantities: [11]

$$M(u(t)) = \int_{\mathbb{R}} u^2(t)dx, \tag{1.2}$$

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}} (\partial_x u)^2(t)dx - \frac{\mu}{k+2}\int_{\mathbb{R}} u^{k+2}(t)dx. \tag{1.3}$$

Next, we describe some previous scattering results for the solutions of (1.1), many of which consider small data scattering problems. That is, when the initial data u_0 is small enough, the corresponding solutions scatter in a certain space. For $k = 3$, Koch and Marzuola [12] proved that the small initial solutions of the focusing gKdV equation ($\mu = 1$) scatter in $\dot{H}^{-1/6}(\mathbb{R})$, relying on refined estimates for a KdV equation linearized at the soliton. As for the case $k = 4$, Kenig, Ponce, and Vega [13], using several linear estimates plus contraction mapping principle, showed that for small initial data in $L^2(\mathbb{R})$, the solutions of the focusing gKdV equation ($\mu = 1$) scatters in $L^2(\mathbb{R})$. And when $k > 4$, Ponce and Vega [14] proved that for small data, solutions of equation $\partial_t u+\partial_{xxx}u+\partial_x(a(u)) = 0$ scatter in $H^1(\mathbb{R})$, in the sense of norm $\|\cdot\|_{s,p}$, by the L^P-decay estimates of the half derivative of the Airy kernel. Kenig, Ponce and Vega [15] also obtained similar scattering results for small initial solutions. Moreover, Farah and Pastor [16] used a new linear estimate to prove that the following conclusion holds. That is, when $u_0 \in \dot{H}^{s_k}(\mathbb{R})$, and satisfies $\|u_0\|_{\dot{H}^{s_k}} \le K$, there exists $\delta = \delta(K)$ such that if $\|U(t)u_0\|_{L_x^{5k/4}L_t^{5k/2}} < \delta$, there is a unique solution Scattering in $\dot{H}^{s_k}(\mathbb{R})$, $s_k = (k-4)/2k$.

After obtaining the small data scattering theory of the global solutions of (1.1), a natural question is whether the global solutions also have scattering results for large initial data, that is, large data scattering problem. However, contrary to the small data scattering theory, only some special cases of the defocusing gKdV equation ($\mu = -1$) have been explicitly proved to have scattering results for large data. For example, when $k = 4$, Dodson [17] showed that when (1.1) are the defocusing gKdV equation ($\mu = -1$), the corresponding solutions are globally well-posed and scattering in $L^2(\mathbb{R})$ for $\forall u_0 \in L^2(\mathbb{R})$, using concentration compactness method. For $k > 4$, the supercritical case, Farah, Linares, Pastor, and Visciglia [18], using a similar approach in [17], proved that when (1.1) is the defocusing supercritical gKdV equation ($\mu = -1$) and k is even, for $\forall u_0 \in H^1(\mathbb{R})$ the correspond solutions are global and scatter in $H^1(\mathbb{R})$. Finally, we should mention that Taegyu Kim [19] recently obtained the conditions for the existence and scattering of global solutions to the subcritical defocusing gKdV equation $\partial_t u + \partial_{xxx} u + \partial_x(|u|^{2\alpha} u) = 0$ in a Morrey space $|\partial_x|^{-\sigma} \hat{M}^{\beta}_{2,\delta}$.

So far, there are many studies on small data scattering problems of (1.1), but most of them are about the focusing gKdV equation ($\mu = 1$), there are relatively few papers considering the defocusing case. Moreover, fewer papers explicitly pointed out how small the initial data should be to yield scattering results for the corresponding global solutions. For these reasons, we decided to further study the small data scattering problem of the global solution of the supercritical gKdV equation.

This paper is mainly inspired by Farah, Linares, Pastor, and Visciglia [18] to discuss the scattering problems of global solutions for the supercritical gKdV equation (1.1), where $k > 4$ is an integer, $u_0 \in H^1(\mathbb{R})$. In this paper, we first establish a scattering criteria, by method like Farah, Linares, Pastor and Visciglia [18]. And then we introduce a new inequality. Next, using the small data theory [16], the inequality introduced before, as well as the conservation of mass (1.2) and energy (1.3), we can obtain the conditions that the global solutions of (1.1) satisfy the scattering criterias. Ultimately, we can obtain the conditions of scattering for the global solutions of the supercritical gKdV equation (1.1) in $H^1(\mathbb{R})$.

The main results of this paper are as follows:

Theorem 1.1. Let $k > 4$, $s_k = (k-4)/2k$, and $u_0 \in H^1(\mathbb{R})$. Assume $\|u_0\|_{L^2} \ne 0$. If u_0 satisfies

$$E(u_0)^{s_k} M(u_0)^{1-s_k} < k^{-\frac{1}{2}} (k-4)^{s_k} 2^{\frac{8-3k}{2k}} (k+2)^{\frac{2}{k}}, \ E(u_0) \ge 0, \tag{1.4}$$

and

$$\|\partial_x u_0\|_{L^2}^{s_k} \|u_0\|_{L^2}^{1-s_k} < k^{-\frac{1}{k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}}, \tag{1.5}$$

then for any t as long as the solution of (1.1) exists, we have

$$\|\partial_x u(t)\|_{L^2}^{s_k} \|u_0\|_{L^2}^{1-s_k} = \|\partial_x u(t)\|_{L^2}^{s_k} \|u(t)\|_{L^2}^{1-s_k} < k^{-\frac{1}{k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}}. \tag{1.6}$$

Theorem 1.2. Let $G=\frac{1}{1-s_k}k^{-\frac{1}{4}}(k-4)^{\frac{k-4}{4k}}2^{\frac{2-k}{2k}}(k+2)^{\frac{1}{k}}$, $k>4$ is an integer, and $u_0\in H^1(\mathbb{R})$ satisfying

$$\|u_0\|_{H^1}\le J,\ J=\min\{G,P(G)/c_3,c_1/c_3\}, \tag{1.7}$$

and

$$E(u_0)\ge 0,\ \max\{(2E(u_0))^{\frac{1}{2}},\|\partial_x u_0\|_{L^2}\}+\|u_0\|_{L^2}<G. \tag{1.8}$$

Then there exists a global solution of (1.1) scattering in both directions, i.e., there exist $\phi_\mu^\pm\in H^1(\mathbb{R})$ such that

$$\lim_{t\to\pm\infty}\|u(t)-U(t)\phi_\mu^\pm\|_{H^1}=0,$$

where $c_1=2^{-1}(4c)^{-1/k}$, $c_2=(2^{-k-3}c^{-2})^{1/(k-1)}$, $P(x)=c_2x^{-1/(k-1)}$, c is a positive constant, and c_3 satisfies

$$\|U(t)u_0\|_{L_x^{5k/4}L_t^{5k/2}}<c_3\|D_x^{s_k}u_0\|_{L_x^2}.$$

The structure of this paper is as follows. Firstly, in section 2, we establish a scattering criteria that make the global solution u of (1.1) scatters in $H^1(\mathbb{R})$, and introduce a small data theory. Then, in section 3, we use the small data theory mentioned before to obtain sufficient conditions for $\|u\|_{L_x^{5k/4}L_t^{5k/2}}<\infty$, which is one of condition of the scattering criteria; Next, in section 4, we introduce an inequality to yield sufficient conditions for u to be uniformly bounded in $H^1(\mathbb{R})$, which is another condition of the scattering criteria. Therefore, in section 5, we can get the scattering results. Finally, we give a brief conclusion in section 6.

Acknowledgement. We would like to thank the reviewers and editors for their careful reading of the manuscript and useful suggestions.

Notation

The notation used in this paper is introduced below.

We use $\|\cdot\|_{L^p}$ to represent the standard $L^p(\mathbb{R})$ norm, and we use subscripts to inform us which variable to focus on. For a given time interval $I\subset R$, define the mixed norms $L_x^pL_t^q$ and $L_x^pL_I^q$ of $f=f(t,x)$ as

$$\|f\|_{L_x^pL_t^q}=\left(\int_{-\infty}^{+\infty}\|f(\cdot,x)\|_{L_t^q}^p\,dx\right)^{1/p},\ \|f\|_{L_x^pL_I^q}=\left(\int_{-\infty}^{+\infty}\|f(\cdot,x)\|_{L_I^q}^p\,dx\right)^{1/p}.$$

Some modifications are needed when $q=\infty$ or $r=\infty$.

We define D_x^s and J_x^s to be, respectively, the Fourier multipliers with symbol $|\xi|^s$ and $\langle\xi\rangle^s=(1+|\xi|)^s$. And the norm in the Sobolev spaces $H^s(\mathbb{R})$ and $\dot{H}^s(\mathbb{R})$ are given, respectively, by

$$\|f\|_{H^s}\equiv\|J^sf\|_{L_x^2}=\left\|\langle\xi\rangle^s\hat{f}\right\|_{L_\xi^2},\quad \|f\|_{\dot{H}^s}\equiv\|D^sf\|_{L_x^2}=\left\||\xi|^s\hat{f}\right\|_{L_\xi^2},$$

where $\hat{f}$ denotes the usual Fourier transform of f.

For any initial data u_0, we use $U(t)u_0$ to denote the solution of the following linear

KdV equation.

$$\begin{cases} \partial_t u + \partial_{xxx} u = 0, \; x \in \mathbb{R}, \; t \in \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases}$$

It is worth noting that $\{U(t)\}_{t=-\infty}^{\infty}$ is a unitary group operator defined in $H^s(\mathbb{R})$ (see [11]).

2 Scattering criteria and small data theory

Proposition 2.1. Let $u_0 \in H^1(\mathbb{R})$, $u(t)$ is a global solution of the following integral equation

$$u(t) = U(t)u_0 - \mu \int_0^t U(t-t')\partial_x (u^{k+1})(t')dt', \tag{2.1}$$

satisfying $\|u\|_{L_x^{5k/4} L_{[0,+\infty)}^{5k/2}} < \infty$ and $\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1(\mathbb{R})} < \infty$, then there exists $\phi_\mu^+ \in H^1(\mathbb{R})$ such that

$$\lim_{t\to+\infty} \|u(t) - U(t)\phi_\mu^+\|_{H^1} = 0. \tag{2.2}$$

Also, if $\|u\|_{L_x^{5k/4} L_{(-\infty,0]}^{5k/2}} < \infty$, then there exists $\phi_\mu^- \in H^1(\mathbb{R})$ such that

$$\lim_{t\to-\infty} \|u(t) - U(t)\phi_\mu^-\|_{H^1} = 0.$$

Proof. The proof of this proposition refers to [18]. However, in [18], only the case of defocusing gKdV ($\mu = -1$)equation is considered. In fact, the same conclusion holds for the focusing gKdV equation ($\mu = 1$).

First prove the defocusing case ($\mu = -1$). Assume $\|u\|_{L_x^{5k/4} L_{[0,+\infty)}^{5k/2}} < \infty$. Let

$$\phi_{-1}^+ = u_0 + \int_0^{+\infty} U(-t')\partial_x (u^{k+1})(t')dt',$$

then we have

$$U(t)\phi_{-1}^+ = U(t)u_0 + \int_0^{+\infty} U(t-t')\partial_x (u^{k+1})(t')dt'.$$

Since u is a global solution of (2.1), we can get

$$u(t) - U(t)\phi_{-1}^+ = -\int_t^{+\infty} U(t-t')\partial_x (u^{k+1})(t')dt'.$$

Through similar analysis as [9], we have

$$\|u(t) - U(t)\phi_{-1}^+\|_{H^1} \le c \|u\|_{L_x^{5k/4} L_{[t,+\infty)}^{5k/2}}^k \left(\|u\|_{L_x^5 L_{[0,+\infty)}^{10}} + \|u_x\|_{L_x^5 L_{[0,+\infty)}^{10}} \right),$$

where c is a positive constant. Since $\|u\|_{L_x^{5k/4} L_{[t,+\infty)}^{5k/2}} \to 0$, when $t \to +\infty$, in order to obtain (2.2), it suffices to verify that

$$\|u\|_{L_x^5 L_{[0,+\infty)}^{10}} + \|u_x\|_{L_x^5 L_{[0,+\infty)}^{10}} < \infty. \tag{2.3}$$

And through analysis, we know that (2.3) is true. Thus, when we take $\mu = -1$, (2.2) holds.

Next, we proof that if we take $\mu = 1$, the same conclusion holds. Let

$$\phi_1^+ = u_0 - \int_0^{+\infty} U(-t')\partial_x (u^{k+1})(t')dt'.$$

Then we have

$$u(t)-U(t)\phi_1^+ = \int_t^{+\infty} U(t-t')\partial_x(u^{k+1})(t')dt'.$$

Through the same analysis as the defocusing case, we can obtain that (2.2) holds too.

In a similar fashion we obtain the second statement. So far, we have proved that as long as a global solution u satisfies $\sup_{t\in\mathbb{R}}\|u(t)\|_{H^1}<\infty$ and $\|u\|_{L_x^{5k/4}L_t^{5k/2}}<\infty$, it scatters in both directions.

Proposition 2.2. Let $k\ge 4$, $s_k=(k-4)/2k$, $u_0\in \dot H^{s_k}(\mathbb{R})$ with $\|u_0\|_{\dot H^{s_k}}\le K<\infty$, I is a time interval. There exists $\delta=\delta(K)$ such that if

$$\|U(t)u_0\|_{L_x^{5k/4}L_t^{5k/2}}<\delta, \tag{2.4}$$

then there is a unique solution u of the integral equation (2.1) in $I\times\mathbb{R}$ with $u\in C(I;\dot H^{s_k}(\mathbb{R}))$ satisfying

$$\|u\|_{L_x^{5k/4}L_t^{5k/2}}\le 2\delta, \text{ and } \|u\|_{L_t^\infty \dot H^{s_k}}+\|D_x^{s_k}u\|_{L_x^5L_t^{10}}<2cK, \tag{2.5}$$

for some positive constant c.

Proof. The proof is quite standard, so only the main idea is shown here. We use the contraction mapping principle to complete our proof, and the detailed proof can be found in [16, Theorem1.2] and [20, Theorem 3.6].

We define $X_{a,b}^K=\left\{u\in C(I;\dot H^{s_k}(\mathbb{R})):\|u\|_{L_x^{5k/4}L_t^{5k/2}}\le a, \|u\|_{L_t^\infty\dot H^{s_k}}+\|D_x^{s_k}u\|_{L_x^5L_t^{10}}<b\right\}$, where $|||u|||=\|u\|_{L_x^{5k/4}L_t^{5k/2}}+\|D_x^{s_k}u\|_{L_x^5L_t^{10}}$, for $\forall u\in X_{a,b}^K$.

On $X_{a,b}^K$ consider the integral operator

$$\Phi(u)(t):=U(t)u_0-\mu\int_0^t U(t-t')\partial_x(u^{k+1})(t')dt'.$$

Then, using the method in [16], we know that when we take $b=2cK$, $ca^k\le 1/2$, $ca^{k-1}b\le 1/2$, $\delta=a/2$, we have

$$\|\Phi(u)\|_{L_x^{5k/4}L_t^{5k/2}}\le a,\ \ \|D_x^{s_k}\Phi(u)(t)\|_{L_t^\infty L_x^2}+\|D_x^{s_k}\Phi(u)(t)\|_{L_x^5L_t^{10}}\le b.$$

Thus, $\Phi: X_{a,b}^K\to X_{a,b}^K$ is well defined.

Next, we need to show that $\Phi(u)$ is a contraction. That is to prove the following two equations hold.

$$|||\Phi(u)-\Phi(v)|||\le\beta|||u-v|||,\ \beta\in(0.1),\ \forall u,v\in X_{a,b}^K.$$

Moreover, from similar arguments in [20], we get

$$\begin{aligned}|||\Phi(u)-\Phi(v)|||&\le c\|D_x^{s_k}(u-v)\|_{L_x^5L_t^{10}}(\|u\|_{L_x^{5k/4}L_t^{5k/2}}^k+\|v\|_{L_x^{5k/4}L_t^{5k/2}}^k)\\&\quad+c\|u-v\|_{L_x^{5k/4}L_t^{5k/2}}(\|u\|_{L_x^{5k/4}L_t^{5k/2}}^{k-1}+\|v\|_{L_x^{5k/4}L_t^{5k/2}}^{k-1})(\|D_x^{s_k}u\|_{L_x^5L_t^{10}}+\|D_x^{s_k}v\|_{L_x^5L_t^{10}})\\&\le 2ca^k\|D_x^{s_k}(u-v)\|_{L_x^5L_t^{10}}+4ca^{k-1}b\|u-v\|_{L_x^{5k/4}L_t^{5k/2}}\end{aligned}$$

Therefore, if we choose $b=2cK$, $\delta=a/2$, $ca^k\le 1/4$, $ca^{k-1}b\le 1/8$, where c is a constant, $\Phi(u)$ is a contraction in $X_{a,b}^K$, satisfying

$$|||\Phi(u)-\Phi(v)|||\le \frac{1}{2}|||u-v|||, \forall u,v\in X_{a,b}^K.$$

Then on the bases of the contraction mapping principle, the proof is complete.

Remark 2.3. Here, δ should satisfy the following range to make the proposition valid. First by $\delta=a/2$, $ca^k\le 1/4$, we can work out the δ should satisfy $\delta\le 2^{-1}(4c)^{-1/k}$, and then from $b=2cK,\ \delta=a/2, ca^{k-1}b\le 1/8$, we have $\delta\le(2^{-k-3}c^{-2})^{1/(k-1)}K^{-1/(k-1)}$. Let's denote $c_1=2^{-1}(4c)^{-1/k}$, $c_2=(2^{-k-3}c^{-2})^{1/(k-1)}$, $P(x)=c_2x^{-1/(k-1)}$, then δ should satisfy

$$\delta\le c_1, \text{ and } \delta\le P(K).$$

3 Sufficient conditions for $\|u\|_{L_x^{5k/4}L_t^{5k/2}}<\infty$

Lemma 3.1. Let $k>4$, $s_k=(k-4)/2k$, then

$$\|U(t)u_0\|_{L_x^{5k/4}L_t^{5k/2}}\le c\|D_x^{s_k}u_0\|_{L_x^2},$$

where c is a positive constant.

Proof. The proof can be found in [18].

Theorem 3.2. Let $k\ge 4$, $s_k=(k-4)/2k$, $u_0\in H^1(\mathbb{R})$, I is a time interval. For $\forall G>0$, assume u_0 satisfies $\|u_0\|_{H^1}\le J$, $J=\min\{G, P(G)/c_3, c_1/c_3\}$. Then there is a unique solution u of (1.1) satisfying $\|u\|_{L_x^{5k/4}L_I^{5k/2}}<2c_3J<\infty$, where $P(x)=c_2x^{-1/(k-1)}$, c_1, c_2 are the constants in Remark 2.3, c_3 is the positive constant in Lemma 3.1.

Proof. We use Proposition 2.2 to complete the proof of the theorem. That is, we need to show that there exists δ satisfying $\delta\le c_1$, and $\delta\le P(J)=c_2J^{-1/(k-1)}$, such that $\|U(t)u_0\|_{L_x^{5k/4}L_I^{5k/2}}<\delta$ in this case.

Next, we discuss it in three cases as follows.

First of all, from Lemma 3.1, we have

$$\|U(t)u_0\|_{L_x^{5k/4}L_I^{5k/2}}\le c_3\|D_x^{s_k}u_0\|_{L_x^2}\le c_3\|u_0\|_{H^1}.$$

Then consider the first case, $J=\min\{G, P(G)/c_3, c_1/c_3\}=G$. From it, one can obtain

$$\begin{cases} G\le P(G)/c_3,\\ G\le c_1/c_3.\end{cases}\Rightarrow \begin{cases} c_3G\le P(G),\\ c_3G\le c_1.\end{cases} \tag{3.1}$$

And when $\|u_0\|_{H^1}\le G$, we have

$$\|U(t)u_0\|_{L_x^{5k/4}L_t^{5k/2}}\le c_3\|D_x^{s_k}u_0\|_{L_x^2}\le c_3\|u_0\|_{H^1}\le c_3G.$$

Therefore, as can be seen from (3.1), taking the value of K, δ in Proposition 2.2 as $K=G=J$, $\delta=c_3G=c_3J$, we have

$$\begin{cases} \delta\le P(K),\\ \delta\le c_1,\\ \|U(t)u_0\|_{L_x^{5k/4}L_t^{5k/2}}\le\delta.\end{cases}$$

That is, (2.4) is true, Proposition 2.2 holds in this case, therefore, from (2.5) we get $\|u\|_{L_x^{5k/4}L_T^{5k/2}} \le 2\delta = 2c_3 J < \infty$.

Next, let's consider the second case, $J = \min\{G, P(G)/c_3, c_1/c_3\} = P(G)/c_3$. From it, we have

$$\begin{cases} P(G)/c_3 \le G, \\ P(G)/c_3 \le c_1/c_3. \end{cases} \Rightarrow \begin{cases} P(G) \le P(P(G)/c_3), \\ P(G) \le c_1. \end{cases} \tag{3.2}$$

The first equation on the right side of (3.2) can be obtained by the monotonic decreasing property of $P(x) = c_2 x^{-1/(k-1)}$, $k > 4$.

And when $\|u_0\|_{H^1} \le P(G)/c_3$, we can obtain

$$\|U(t)u_0\|_{L_x^{5k/4}L_t^{5k/2}} \le c_3 \|D_x^{s_k} u_0\|_{L_x^2} \le c_3 \|u_0\|_{H^1} \le c_3 (P(G)/c_3) = P(G).$$

Therefore, as can be seen from (3.2), taking the value of K, δ in Proposition 2.2 as $K = P(G)/c_3 = J$, $\delta = P(G) = c_3 J$, we have

$$\begin{cases} \delta \le P(K), \\ \delta \le c_1, \\ \|U(t)u_0\|_{L_x^{5k/4}L_t^{5k/2}} \le \delta. \end{cases}$$

That is, (2.4) is also true, Proposition 2.2 holds too, therefore, from (2.5) we get $\|u\|_{L_x^{5k/4}L_T^{5k/2}} \le 2\delta = 2c_3 J < \infty$ in the second case.

Finally, consider the third case, $J = \min\{G, P(G)/c_3, c_1/c_3\} = c_1/c_3$. From analysis similar to the second case, we have

$$\begin{cases} c_1/c_3 \le G, \\ c_1/c_3 \le P(G)/c_3. \end{cases} \Rightarrow \begin{cases} P(G) \le P(c_1/c_3), \\ c_1 \le P(G). \end{cases} \tag{3.3}$$

And when $\|u_0\|_{H^1} \le c_1/c_3$, we can obtain

$$\|U(t)u_0\|_{L_x^{5k/4}L_t^{5k/2}} \le c_3 \|D_x^{s_k} u_0\|_{L_x^2} \le c_3 \|u_0\|_{H^1} \le c_3 (c_1/c_3) = c_1.$$

Thus, by (3.3), taking the value of K, δ in Proposition 2.2 as $K = c_1/c_3 = J$, $\delta = c_1 = c_3 J$, we can get

$$\begin{cases} \delta \le P(K), \\ \delta \le c_1, \\ \|U(t)u_0\|_{L_x^{5k/4}L_t^{5k/2}} \le \delta. \end{cases}$$

Similarly, Proposition 2.2 holds in the third case, that is $\|u\|_{L_x^{5k/4}L_T^{5k/2}} \le 2\delta = 2c_3 J < \infty$.

All in all, we can get that for $\forall G > 0$, if u_0 satisfies $\|u_0\|_{H^1} \le J$, where $J = \min\{G, P(G)/c_3, c_1/c_3\}$, we can take $K = J$, $\delta = c_3 J$ such that Proposition 2.2 holds. That is to say, there is a unique u of (1.1) satisfying $\|u\|_{L_x^{5k/4}L_T^{5k/2}} < 2c_3 J < \infty$.

4 Sufficient conditions for uniformly bounded solutions

Lemma 4.1. Assume $u \in W^1(\mathbb{R})$, and $u \in L^p(\mathbb{R})$, $u' \in L^r(\mathbb{R})$, where $p, r \in [1, \infty]$.

Then for $\forall q \in [p, \infty]$, we have $u \in L^q(\mathbb{R})$ and

$$\| u\|_{L^q} \le \theta^{-(1-\frac{p}{q})\theta} \| u\|_{L^p}^{1-(1-\frac{p}{q})\theta} \| u'\|_{L^r}^{(1-\frac{p}{q})\theta}, \tag{4.1}$$

where $\theta = r/(pr+r-p)$, $W^1(\mathbb{R})$ represents a space consisting of all 1^{th} Weakly differentiable functions.

Proof. See Exercise 1.8 in [21] and Introduction of [22]. A detailed proof of this lemma is given below.

Define $W^{1,p,r}(\mathbb{R}) = \{u : u \in L^p(\mathbb{R}),\ u' \in L^r(\mathbb{R})\}$. And let $W_0^{1,p,r}(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ in $W^{1,p,r}(\mathbb{R})$, one can show that $W_0^{1,p,r}(\mathbb{R}) = W^{1,p,r}(\mathbb{R})$. Moreover, to show that (4.1) holds, it means that the following two statements hold for $\forall u \in W^{1,p,r}(\mathbb{R})$.

$$\sup_{x\in\mathbb{R}}|u| \le \theta^{-\theta}(\int_{\mathbb{R}}|u|^p\, dx)^{\frac{1-\theta}{p}}(\int_{\mathbb{R}}|u'|^r\, dx)^{\frac{\theta}{r}}, \qquad q = \infty. \tag{4.2}$$

$$\int_{\mathbb{R}}|u|^q\, dx \le \theta^{-(q-p)\theta}(\int_{\mathbb{R}}|u|^p\, dx)^{\frac{q-(q-p)\theta}{p}}(\int_{\mathbb{R}}|u'|^r\, dx)^{\frac{(q-p)\theta}{r}}, \quad p \le q < \infty. \tag{4.3}$$

And from $W_0^{1,p,r}(\mathbb{R}) = W^{1,p,r}(\mathbb{R})$, we only need to prove that for $\forall u \in W_0^{1,p,r}(\mathbb{R})$, (4.2) and (4.3) hold.

First of all, it's easy to know

$$u(x) = \int_{-\infty}^{x} u'(t)dt = -\int_{x}^{\infty} u'(t)dt, \quad u \in W_0^{1,p,r}(\mathbb{R}).$$

Thus,

$$|u(x)| \le \int_{x}^{\infty}|u'(t)|\, dt \le \int_{-\infty}^{\infty}|u'(t)|\, dt, \quad u \in W_0^{1,p,r}(\mathbb{R}). \tag{4.4}$$

Secondly, we can prove that

$$\frac{d}{dx}|u|^\alpha = \alpha|u|^{\alpha-2}uu'. \tag{4.5}$$

$$\frac{d}{dx}|u|^\alpha = \frac{d}{dx}(u^2)^{\frac{\alpha}{2}} = \frac{\alpha}{2}(u^2)^{\frac{\alpha}{2}-1}2uu'$$
$$= \alpha(u^2)^{\frac{\alpha}{2}-1}uu'$$
$$= \alpha|u|^{\alpha-2}uu'.$$

Therefore, taking $u(x)$ in (4.4) as $|u|^\alpha$, and using (4.5) together with Hölder's inequality, we can get

$$|u|^\alpha \le \int_{\mathbb{R}}\left|\frac{d}{dt}|u|^\alpha\right|dt = \alpha\int_{\mathbb{R}}|u|^{\alpha-2}|uu'|\, dt$$
$$= \alpha\int_{\mathbb{R}}|u|^{\alpha-1}|u'|\, dt$$
$$\le \alpha(\int_{\mathbb{R}}|u|^{(\alpha-1)(1-\frac{1}{r})^{-1}}\, dx)^{1-\frac{1}{r}}(\int_{\mathbb{R}}|u'|^r\, dx)^{\frac{1}{r}}.$$

In order to construct $\|u\|_{L^p}$ on the right-hand side of (4.1), we set $(\alpha-1)(1-\frac{1}{r})^{-1} = p$, which yields $\alpha = (pr+r-p)/r$. Then denote $\alpha = \frac{1}{\theta}$. We have

$$|u|^{\frac{1}{\theta}} \le \frac{1}{\theta}(\int_{\mathbb{R}}|u|^p\, dx)^{1-\frac{1}{r}}(\int_{\mathbb{R}}|u'|^r\, dx)^{\frac{1}{r}}.$$

Thus, we can obtain

$$\sup_{x\in\mathbb{R}}|u|^{\frac{1}{\theta}} \le \frac{1}{\theta}(\int_{\mathbb{R}}|u|^p\,dx)^{1-\frac{1}{r}}(\int_{\mathbb{R}}|u'|^r\,dx)^{\frac{1}{r}}\ . \tag{4.6}$$

On the one hand, consider $q=\infty$, by (4.6)

$$\sup_{x\in\mathbb{R}}|u| = \sup_{x\in\mathbb{R}}(|u|^{\frac{1}{\theta}})^{\theta} \le (\frac{1}{\theta})^{\theta}(\int_{\mathbb{R}}|u|^p\,dx)^{(1-\frac{1}{r})\theta}(\int_{\mathbb{R}}|u'|^r\,dx)^{\frac{\theta}{r}}$$

$$=(\frac{1}{\theta})^{\theta}(\int_{\mathbb{R}}|u|^p\,dx)^{\frac{1-\theta}{p}}(\int_{\mathbb{R}}|u'|^r\,dx)^{\frac{\theta}{r}}.$$

Here, the last equation makes use of the relation $(\frac{1}{\theta}-1)(1-\frac{1}{r})^{-1}=p$, from which we know that $(1-\frac{1}{r})\theta=\frac{1-\theta}{p}$.

On the other hand, when $p\le q<\infty$, similarly, we can obtain

$$\int_{\mathbb{R}}|u|^q dx \le \sup_{x\in\mathbb{R}}|u|^{q-p}\int_{\mathbb{R}}|u|^p dx = \sup_{x\in\mathbb{R}}(|u|^{\frac{1}{\theta}})^{(q-p)\theta}\int_{\mathbb{R}}|u|^p dx$$

$$\le (\frac{1}{\theta})^{(q-p)\theta}(\int_{\mathbb{R}}|u|^p\,dx)^{(1-\frac{1}{r})(q-p)\theta}(\int_{\mathbb{R}}|u'|^r\,dx)^{\frac{(q-p)\theta}{r}}\int_{\mathbb{R}}|u|^p dx$$

$$=(\frac{1}{\theta})^{(q-p)\theta}(\int_{\mathbb{R}}|u|^p\,dx)^{(1-\frac{1}{r})(q-p)\theta+1}(\int_{\mathbb{R}}|u'|^r\,dx)^{\frac{(q-p)\theta}{r}}.$$

From the previous analysis, we know that $(1-\frac{1}{r})\theta=\frac{1-\theta}{p}$. Thus, we can get that for $\forall u\in W_0^{1,p,r}(\mathbb{R})$, (4.2), (4.3) hold. Lemma is proved.

Proof of Theorem 1.1. The proof is inspired by the works of [23]. Firstly, let's assume u belongs to $H^1(\mathbb{R})$, then we have $u\in L^2(\mathbb{R}),\ u'\in L^2(\mathbb{R})$. Therefore, applying Lemma 4.1, with $p=r=2$, $q=k+2$, we can deduce

$$\|u\|_{L^{k+2}} \le 2^{\frac{k}{2(k+2)}}\|u\|_{L^2}^{\frac{k+4}{2(k+2)}}\|u'\|_{L^2}^{\frac{k}{2(k+2)}}.$$

Taking $k+2$ power on both sides at the same time, we can get,

$$\|u\|_{L^{k+2}}^{k+2} \le 2^{\frac{k}{2}}\|u\|_{L^2}^{\frac{k+4}{2}}\|u'\|_{L^2}^{\frac{k}{2}}. \tag{4.7}$$

Next, let u be the solution of (1.1) on $I\times\mathbb{R}$, I is a time interval. Then, using the conserved quantities (1.2), (1.3) and the above equation, we obtain

$$\begin{aligned}\|\partial_x u(t)\|_{L^2}^2 &= 2E(u_0)+\frac{2\mu}{k+2}\int_{\mathbb{R}}u^{k+2}(t)dx\\ &\le 2E(u_0)+\frac{2}{k+2}\|u\|_{L^{k+2}}^{k+2}\\ &\le 2E(u_0)+\frac{2}{k+2}2^{\frac{k}{2}}\|u_0\|_{L^2}^{\frac{k+4}{2}}\|\partial_x u(t)\|_{L^2}^{\frac{k}{2}}.\end{aligned} \tag{4.8}$$

When $\|u_0\|_{L^2}\ne 0$, let $X(t)=\|\partial_x u(t)\|_{L^2}^2$, $A=2E(u_0)$, $B=\frac{2}{k+2}2^{\frac{k}{2}}\|u_0\|_{L^2}^{\frac{k+4}{2}}$, then (4.8) can be rewritten to

$$X(t)-BX(t)^{\frac{k}{4}} \le A, t\in I.$$

Let $f(x)=x-Bx^{k/4}$, for $x\geq 0$. By calculation, it is easy to obtain that $f(x)$ is monotonically increasing in $[0,x_0)$ and monotonically decreasing in $(x_0,+\infty)$. Moreover, the function f has a local maximum at $x_0=(\frac{4}{kB})^{4/(k-4)}$ with maximum $f(x_0)=\frac{k-4}{k}(\frac{4}{kB})^{4/(k-4)}$. Thus, if we require

$$A=2E(u_0)<f(x_0),\ X(0)<x_0, \tag{4.9}$$

the continuity of $X(t)$ implies that $X(t)<x_0$ for $\forall t$, whenever the solution exists.

Finally, let's calculate the specific value of $f(x_0)$, x_0 to turn (4.9) into conditions (1.4), (1.5).

First substituting $B=\frac{2}{k+2}2^{\frac{k}{2}}\|u_0\|_{L^2}^{\frac{k+4}{2}}$ into $f(x_0)$, we have

$$f(x_0)=\frac{k-4}{k}(\frac{4}{kB})^{\frac{4}{k-4}}=\frac{k-4}{k}(\frac{4}{k\frac{2}{k+2}2^{\frac{k}{2}}\|u_0\|_{L^2}^{\frac{k+4}{2}}})^{\frac{4}{k-4}}$$

$$=\frac{(k-4)2^{\frac{2(2-k)}{k-4}}(k+2)^{\frac{4}{k-4}}}{k^{\frac{k}{k-4}}\|u_0\|_{L^2}^{\frac{2(k+4)}{k-4}}}.$$

Therefore,

$$\frac{1}{2}f(x_0)=\frac{(k-4)2^{\frac{8-3k}{k-4}}(k+2)^{\frac{4}{k-4}}}{k^{\frac{k}{k-4}}\|u_0\|_{L^2}^{\frac{2(k+4)}{k-4}}}.$$

Then, taking s_k power on both sides at the same time, we can get

$$E(u_0)^{s_k}<(\frac{(k-4)2^{\frac{8-3k}{k-4}}(k+2)^{\frac{4}{k-4}}}{k^{\frac{k}{k-4}}\|u_0\|_{L^2}^{\frac{2(k+4)}{k-4}}})^{s_k}=\frac{(k-4)^{s_k}2^{\frac{8-3k}{2k}}(k+2)^{\frac{2}{k}}}{k^{\frac{1}{2}}\|u_0\|_{L^2}^{\frac{2(k+4)}{2k}}}.$$

And $1-s_k=(k+4)/2k$, so we can obtain the first condition of (1.4).

Similarly, from

$$x_0=(\frac{4}{kB})^{4/(k-4)}=\frac{2^{\frac{2(2-k)}{k-4}}(k+2)^{\frac{4}{k-4}}}{k^{\frac{4}{k-4}}\|u_0\|_{L^2}^{\frac{2(k+4)}{k-4}}},$$

and the second condition of (4.9), we can obtain (1.5). By the same method, it can be proved that (1.6) holds. Thus, theorem is proved.

Remark 4.2. For the defocusing gKdV equation $(\mu=-1)$, when $k>4$ is an even, obviously we can obtain $E(u_0)\geq 0$, then if we know $\|u_0\|_{L^2}\neq 0$, as long as u_0 satisfies (1.4), (1.5), we can get the conclusion.

Moreover, from the conclusion, we know that $\|\partial_x u(t)\|_{L^2}$ of the solution u is bounded on I. Then by the definition of the H^1-norm, and the conservation of mass (1.2), we can get that the H^1-norm of u is also bounded. That is, u is a uniformly bounded solution.

Lemma 4.3. Assume $a,b \ge 0, c \in (0,1)$, then

$$a^c b^{1-c} \le ca + (1-c)b.$$

Proof. If $a=0$ or $b=0$, the conclusion clearly holds. Otherwise, if $a>0$ and $b>0$, let's take the natural logarithm on both sides of the above inequality. It's equivalent to show

$$c\ln a + (1-c)\ln b \le \ln(ca+(1-c)b).$$

And the above formula obviously holds, due to the property of $\ln(x)$. Thus, the lemma is proved.

Corollary 4.4. Let $u_0 \in H^1(\mathbb{R})$, $k>4$ is an integer, $s_k = (k-4)/2k$. Assume $E(u_0) \ge 0$, $\|u_0\|_{L^2} \ne 0$ and

$$\max\{(2E(u_0))^{\frac{1}{2}}, \|\partial_x u_0\|_{L^2}\} + \|u_0\|_{L^2} < \frac{1}{1-s_k} k^{-\frac{1}{4}}(k-4)^{\frac{k-4}{4k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}}. \quad (4.10)$$

Then, Theorem 1.1 holds too, that is, for any t as long as the solution exists,

$$\|\partial_x u(t)\|_{L^2}^{s_k} \|u_0\|_{L^2}^{1-s_k} < k^{-\frac{1}{k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}}.$$

Proof. If we want to show that Theorem 1.1 holds according to the conditions in the corollary, we need to prove that u_0 satisfies (1.4) and (1.5).

Firstly, let's consider the first case, $\max\{(2E(u_0))^{\frac{1}{2}}, \|\partial_x u_0\|_{L^2}\} = (2E(u_0))^{\frac{1}{2}}$. If u_0 satisfies this condition, we have $\|\partial_x u_0\|_{L^2} \le (2E(u_0))^{\frac{1}{2}}$. Then (4.10) can be written as

$$(2E(u_0))^{\frac{1}{2}} + (M(u_0))^{\frac{1}{2}} < \frac{1}{1-s_k} k^{-\frac{1}{4}}(k-4)^{\frac{k-4}{4k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}}.$$

On the one hand, multiplying both sides by $1-s_k$, we can obtain

$$(1-s_k)((2E(u_0))^{\frac{1}{2}} + (M(u_0))^{\frac{1}{2}}) < k^{-\frac{1}{4}}(k-4)^{\frac{k-4}{4k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}}.$$

Moreover, from $s_k < 1-s_k$, we have

$$s_k(2E(u_0))^{\frac{1}{2}} + (1-s_k)(M(u_0))^{\frac{1}{2}} < k^{-\frac{1}{4}}(k-4)^{\frac{k-4}{4k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}}.$$

Now, using Lemma 4.3, we get

$$(2E(u_0))^{\frac{s_k}{2}} (M(u_0))^{\frac{1-s_k}{2}} \le s_k(2E(u_0))^{\frac{1}{2}} + (1-s_k)(M(u_0))^{\frac{1}{2}}.$$

Thus, from the above two equations, we have

$$(2E(u_0))^{\frac{s_k}{2}} (M(u_0))^{\frac{1-s_k}{2}} < k^{-\frac{1}{4}}(k-4)^{\frac{k-4}{4k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}}. \quad (4.11)$$

It's clear that (4.11) is equivalent to (1.4).

On the other hand, from $\|\partial_x u_0\|_{L^2} \le (2E(u_0))^{\frac{1}{2}}$ and (4.11), we have

$$\|\partial_x u(t)\|_{L^2}^{s_k} \|u_0\|_{L^2}^{1-s_k} < k^{-\frac{1}{4}}(k-4)^{\frac{k-4}{4k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}}.$$

And for $k>4$, we have $k^{-\frac{1}{4}}(k-4)^{\frac{k-4}{4k}} < k^{-\frac{1}{k}}$, thus, there is

$$\|\partial_x u_0\|_{L^2}^{s_k} \|u_0\|_{L^2}^{1-s_k} < k^{-\frac{1}{k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}}.$$

That is (1.5).

Therefore, according to (4.11), and the above equation, the conclusion holds in the first case.

Now let's consider the second case, $\max\{(2E(u_0))^{\frac{1}{2}}, \|\partial_x u_0\|_{L^2}\} = \|\partial_x u_0\|_{L^2}$, that is $(2E(u_0))^{\frac{1}{2}} < \|\partial_x u_0\|_{L^2}$. Thus, (4.10) can be rewritten as

$$\|\partial_x u_0\|_{L^2} + \|u_0\|_{L^2} < \frac{1}{1-s_k} k^{-\frac{1}{4}}(k-4)^{\frac{k-4}{4k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}}.$$

In a similar way, we can get

$$\|\partial_x u_0\|_{L^2}^{s_k} \|u_0\|_{L^2}^{1-s_k} < k^{-\frac{1}{4}}(k-4)^{\frac{k-4}{4k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}}. \tag{4.12}$$

Similarly, by $k^{-\frac{1}{4}}(k-4)^{\frac{k-4}{4k}} < k^{-\frac{1}{k}}$, we have

$$\|\partial_x u_0\|_{L^2}^{s_k} \|u_0\|_{L^2}^{1-s_k} < k^{-\frac{1}{k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}},$$

that is (1.5). Moreover, due to $(2E(u_0))^{\frac{1}{2}} < \|\partial_x u_0\|_{L^2}$ and (4.12), we can obtain

$$(2E(u_0))^{\frac{s_k}{2}} \|u_0\|_{L^2}^{1-s_k} < k^{-\frac{1}{4}}(k-4)^{\frac{k-4}{4k}} 2^{\frac{2-k}{2k}} (k+2)^{\frac{1}{k}},$$

which is equivalent to (1.4). Thus, we complete the proof of this corollary.

5 Scattering result

Proof of Theorem 1.2. Our aim is to prove that when u_0 satisfies (1.7), (1.8), there exists a global solution of (1.1) scattering in $H^1(\mathbb{R})$. By Proposition 2.1, Theorem 3.2, Remark 4.2, Corollary 4.4, the conclusion is obviously valid. Specific analysis is as follows.

Firstly, from Proposition 2.1, we know that we need to show that there exists a global solution u satisfying $\|u\|_{L_x^{5k/4} L_t^{5k/2}} < \infty$ and $\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} < \infty$.

Secondly, it follows from Theorem 3.2 that if u_0 satisfies (1.7), then (1.1) has a global solution u on $\mathbb{R}\times\mathbb{R}$ with $\|u\|_{L_x^{5k/4} L_t^{5k/2}} < \infty$. Thus, we get a global solution u satisfying the first condition.

Finally, from Remark 4.2 and Corollary 4.4 we can get that if u_0 satisfies (1.7) as well as (1.8), not only does the global solution u satisfy $\|u\|_{L_x^{5k/4} L_t^{5k/2}} < \infty$, but also $\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} < \infty$. Thus, theorem is proved.

Remark 5.1. Although it can be seen from Proposition 2.1 that as long as the global solution u of (1.1) satisfies $\|u\|_{L_x^{5k/4} L_t^{5k/2}} < \infty$ and $\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} < \infty$, we can obtain that the solution u scatters in $H^1(\mathbb{R})$. However, so far, only Farah et al [18] has shown that when $k > 4$ is even and $\mu = -1$, for $\forall u_0 \in H^1(\mathbb{R})$, the corresponding solution of (1.1) is global and scattering (large data scattering). For other cases of large data scattering problems of the supercritical gKdV equation, for example, the defocusing gKdV equation ($\mu = -1$) when $k > 4$ is odd or the focusing gKdV equation ($\mu = 1$) when $k > 4$ is an integer, whether the corresponding global solutions are scattering are not yet proven. Although these problems are more difficult, they are also very worthy of study.

6 Conclusion

In this paper, we consider solutions of the supercritical generalized KdV equation $\partial_t u+\partial_{xxx}u+\mu\partial_x(u^{k+1}) = 0$, where $k > 4$ is an integer and $\mu = \pm 1$. For any u_0 belongs to $H^1(\mathbb{R})$, we show that if $u_0 \neq 0$ satisfies conditions in Theorem 1.2, then the equation (1.1) has a global solution which scatters in $H^1(\mathbb{R})$.

In the study of scattering problem, this paper first establishes a scattering criteria. Then we creatively introduce an inequality to obtain the conditions that make the equation have uniformly bounded solutions in $H^1(\mathbb{R})$. This result is more specific than the one in [23]. Finally, we further clarify the conditions for the equation to have a global solution scattering in $H^1(\mathbb{R})$. Moreover, the scattering results we obtain are applicable not only to the defocusing case, but also to the focusing case.

However, as we mentioned before, for large data scattering problems, only Farah et al [18] has shown that when $k > 4$ is even and $\mu = -1$, for any u_0 belongs to $H^1(\mathbb{R})$, the corresponding solution of (1.1) is global and scattering, it is not clear whether the global solutions in other cases also scatter in $H^1(\mathbb{R})$.

Authors' contribution

This work was completed in collaboration between both authors. Both authors read and approved the final manuscript.

References

[1] Kenig CE, Merle F. Global well-posedness, scattering, and blow-up for the energy-critical focusing nonlinear Schrödinger equation in the radial case. Invent. Math. 2006; 166: 645–675.

[2] Holmer J, Roudenko S. A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation, Commun. Math. Phys. 2008; 282: 435–467.

[3] Duyckaerts T, Holmer J, Roudenko S. Scattering for the non-radial 3D cubic nonlinear Schrödinger equations. Math. Res. Lett. 2008; 15: 1233–1250.

[4] Fang D, Xie J, Cazenave T. Scattering for the focusing, energy-subcritical nonlinear Schrödinger equation. Sci. China Math. 2011; 54: 2037-2062.

[5] Kopçasız B, Yaşar E. Novel exact solutions and bifurcation analysis to dual-mode nonlinear Schrödinger equation. J. Ocean Eng. Sci. 2022.

[6] Kopçasız B, Yaşar E. Exploration of interactional phenomena and multi-wave solutions of the fractional-order dual-mode nonlinear Schrödinger equation. Math. Meth. Appl. Sci. 2023; 1–19.

[7] Kopçasız B, Seadawy AR, Yaşar E. Highly dispersive optical soliton molecules to dual-mode nonlinear Schrödinger wave equation in cubic law media. Opt Quant Electron. 2022; 54: 194.
[8] Kenig CE, Merle F. Global well-posedness, scattering, and blow-up for the energy-critical focusing nonlinear wave equation. Acta Math. 2008; 201: 147–212.
[9] Dodson B, Lawrie A, Mendelson D, Murphy J. Scattering for defocusing energy subcritical nonlinear wave equations. Analysis & PDE. 2020; 13: 1995-2090.
[10] Miura RM, Gardner CS, Kruskal MD. Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion. J. Math. Phys. 1968; 9(8): 1204-1209.
[11] Linares F, Ponce G. Introduction to Nonlinear Dispersive Equations. Springer Verlag: New York. 2009.
[12] Koch H, Marzuola JL. Small data scattering and soliton stability in $\dot{H}^{-1/6}$ for the quartic KdV equation. Anal. PDE. 2012; 5: 145-198.
[13] Kenig CE, Ponce G, Vega L. Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. Comm. Pure Appl. Math. 1993; 46: 527-620.
[14] Ponce G, Vega L. Nonlinear Small Data Scattering for the Generalized Korteweg-de Vries Equation. J. Funct. Ana. 1990; 90: 445-457.
[15] Kenig CE, Ponce G, Vega L. Oscillatory integrals and regularity of dispersive equations. Indiana Univ. Math. J. 1991; 40: 33-69.
[16] Farah LG, Pastor A. On Well-posedness and wave operator for the gKdV equation. Bull. Sci. Math. 2013;137: 229-241.
[17] Dodson B. Global well-posedness and scattering for the defocusing, mass-critical generalized KdV equation. Ann. PDE. 2017; 3: 5.
[18] Farah LG, Linares F, Pastor A, Visciglia N. Large data scattering for the defocusing supercritical generalized KdV equation. Communications in Partial Differential Equations. 2018; 43: 118-157.
[19] Taegyu Kim. On the scattering of subcritical defocusing generalized Korteweg-de Vries equation. Discrete and Continuous Dynamical Systems. 2023; 43(6): 2241-2269.
[20] Farah LG, Pigott B. Nonlinear Profile Decomposition and the Concentration Phenomenon for Supercritical Generalized KdV Equations. Indiana University Mathematics Journal. 2018; 67(5): 1857-1892.
[21] Shangbin Cui. Introduction to Modern Theory of Partial Differential Equations (Chinese Edition). Beijing: Science Press. 2015.
[22] Petersson JH. Best constants for Gagliardo-Nirenberg inequalities on the real line. Nonlinear Anal. 2007; 67: 587-600.
[23] Farah LG, Linares F, Pastor A. The supercritical generalized KdV equation: Global well-posedness in the energy space and below. Math. Res. Lett. 2011; 18: 357-377.